Linear Algebra Final Exam

Fall 2007

Name: ________________________________

Soc. Sec. No.: ________________________________

Signature: ________________________________

Note: no credit will be given if your work is not shown!

For the following three problems,

\[
A = \begin{bmatrix}
-1 & 2 & -1 & 0 \\
1 & 0 & -1 & 2 \\
1 & -2 & 3 & -4
\end{bmatrix}
\]

1. Find a basis for the nullspace of \( A \).

2. Find an orthogonal basis for the orthogonal complement of the row space of \( A \).

3. Let \( \vec{b} = (1, -3, a)^t \). Give conditions on \( a \) so that \( A\vec{x} = \vec{b} \) has

(a). a unique solution.

(b). more than one solution.

(c). no solution.
4. Let

\[
A = \begin{bmatrix}
-3 & 0 & 0 \\
2 & -4 & 0 \\
2 & -1 & -4 \\
\end{bmatrix}
\]

(a). Find the trace of \( A \).

(b). Find the characteristic polynomial of \( A \).

(c). Find all of the eigenvalues for \( A \).

(d). For each eigenvalue of \( A \), find a basis for its eigenspace.

(e). For each eigenvalue of \( A \), find both of its algebraic and geometric multiplicities.

(f). Is \( A \) diagonalizable? If \( A \) is diagonalizable, then find a matrix \( S \) and a diagonal matrix \( D \) such that \( A = SDS^{-1} \). If \( A \) is not diagonalizable, explain why?
5. Let $V$ be the 4-dimensional vector space of polynomials of degree three (3) or less.

   Consider the linear transformation $L : V \to V$ defined by $L(p) = xp'(x) - 2p(x)$.

(a). Find the matrix $A$ which represents $L$ in the standard basis $\{1, x, x^2, x^3\}$ for $V$.

(b). Find a basis for the nullspace of $L$.

(c). Find a basis for the range space of $L$.

(d). What is the rank of $L$?

(e). What is the nullity of $L$?

(f). What is the rank of $A$?

(g). What is the dimension of the nullspace of $A$?

(h). Find the characteristic polynomial for $L$.

(i). Find all of the eigenvalues for $L$.

(j). Is $L$ diagonalizable? If $L$ is diagonalizable, then find a basis $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)$ for $V$ and a diagonal matrix $D$ such that $L(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4) = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)D$. If $L$ is not diagonalizable, explain why?
6. Let

\[ A = \begin{bmatrix}
  1 & 3 & 2 \\
  2 & 1 & 0 \\
  4 & 5 & 1 \\
\end{bmatrix} \]

(a). Find \( \det(A) \).

(b). If \( A \) is nonsingular, find its inverse. If \( A \) is singular, explain why?

(c). Find its (3, 1)-cofactor \( A_{31} \) for \( A \).

(d). Find the product matrix \( A(\text{adj}A) \).

7. Let \( A \) be an \( m \times n \) matrix of real numbers. If the dimension of the nullspace of \( A \) is \( k \), compute the following in terms of \( k \) and the dimension of \( A \).

(a). \( \dim \mathcal{R}(A) \).

(b). \( \dim \mathcal{R}(A^t) \).

(c). \( \dim \mathcal{N}(A) \).

(d). \( \dim \mathcal{N}(A^t) \).

(e). Find an orthogonal decomposition for both \( \mathbb{R}^m \) and \( \mathbb{R}^n \) in terms of the four fundamental subspaces associated with \( A \).
8. Let \( \bar{u} \) be a given column vector in \( \mathbb{R}^n \) and \( |\bar{u}|^2 = 2 \). Consider the matrix \( R = I - \bar{u}\bar{u}^T \).

(a). Show that \( R \) is an orthogonal matrix.

(b). Compute \( R\bar{u} \).

(c). If \( \bar{v} \) is orthogonal to \( \bar{u} \), compute \( R\bar{v} \).

(d). Find all of the eigenvalues for the matrix \( R \) and their corresponding algebraic and geometric multiplicities.

9. Let \( L : R^3 \rightarrow R^3 \) be the linear transformation defined by the reflection through the plane \( P = \{(x, y, z) \in R^3 | x - y = 0 \} \).

(a). Find an orthonormal basis for \( P \).

(b). Find an orthonormal basis for \( P^\perp \).

(c). Find an orthonormal basis such that the matrix representation of \( L \) in this orthonormal basis is a diagonal matrix.

(d). Find the matrix \( A \) that represents \( L \) with respect to the standard basis for \( R^3 \).
(10). Let $A$ and $B$ be two $n \times n$ matrices.

(a). Define $A$ to be similar to $B$.

(b). Define $A$ to be congruent to $B$.

(c). Is a real symmetric matrix always similar to a diagonal matrix? Why?

(d). Is a real symmetric matrix always congruent to a diagonal matrix? Why?

11. Let $\vec{z} = (x, y)^t$ and $Q(\vec{z}) = 23x^2 - 72xy + 2y^2$.

(a). Find the symmetric bilinear form $B$ associated with the quadratic form $Q$.

(b). Find a symmetric matrix $A$ such that $Q(\vec{z}) = \vec{z}^t A \vec{z}$.

(c). Find an orthogonal matrix $S$ such that $A$ is congruent to a diagonal matrix $D$ via $S$.

(d). Use the principal axes theorem to diagonalize the quadratic form $Q$.

(e). Does $Q$ has a local extrema at the origin? Explain.

12. State the spectral theorem for a self-adjoint linear transformation from a finite dimensional real product space to itself.
13. Let

\[
A = \begin{bmatrix}
-2 & 1 & 0 & -1 \\
1 & 0 & -1 & 2 \\
0 & -1 & 2 & -3
\end{bmatrix}
\]

(a). Find a basis for each of the four fundamental subspaces associated with \( A \).

(b). Find the rank of \( A \), the nullity of \( A \), and the nullity of \( A^t \).

(c). Find the dimension of the row space of \( A \) and the dimension of the column space of \( A \).

(d). Are the three row vectors of \( A \) linearly independent? Justify your answer by either prove they are linearly independent or provide an explicit nontrivial linear relation of the three row vectors.

(e). Is \( \mathcal{N}(A) \) isomorphic to \( \mathcal{R}(A) \)? Is \( \mathcal{R}(A) \) isomorphic to \( \mathcal{R}(A^t) \)? Is \( \mathcal{N}(A) \) isomorphic to \( \mathcal{N}(A^t) \)?