(1) (1 point) True or False: Bendixson’s first name was PeeWee.

(2) Using a Lyapunov function of the form $V = ax^2 + cy^2$, show that the origin is a stable equilibrium point of the system $x' = -x^3 + 2y^3$, $y' = -2xy^2$.

(3) Write down the equation for a spring with linear restoring force, damping (small, subcritical) proportional to velocity. Explain your notation. Rewrite as a $2 \times 2$ first-order system. Describe the phase portrait of this system.

(4) Two point masses in the plane move according to the gravitational attraction between them (i.e., inverse-square law; don’t worry about the correct physical constants of proportionality). Formulate as an $n \times n$ first-order system. (What is $n$?)

(5) Use Lyapunov function of form $c(x^2 + y^2)$ to show that the origin is an asymptotically stable equilibrium point of $x' = y - xf(x^2 + y^2)$, $y' = -x - yf(x^2 + y^2)$, where $f(0) = 0$ and $f > 0$ otherwise. Write down the linearization of this system at the origin. With respect to the linearized system is the origin asymptotically stable?

(6) Let $H$ be a smooth function of two variables. Show that $H(x, y)$ is conserved (not changing with time) under the flow of the system $x' = \partial H/\partial y$, $y' = -\partial H/\partial x$.

(7) State the Poincaré-Bendixson theorem. Use it to show that the system $x' = -2y$, $y' = 3x$ has a cycle about the origin.

(8) Prove: Let $A$ be a $2 \times 2$ constant matrix. If the trace of $A$ (the sum of the diagonal elements) is not zero, then the $2 \times 2$ system $\mathbf{x}' = A \mathbf{x}$ has no cycles.

(9) True or false (explain). Consider the nonlinear system SYSTEM1: $\mathbf{x}' = g(\mathbf{x})$ with $g(0) = 0$. Let $A$ be the Jacobian matrix of $g$ at 0, and consider the system SYSTEM2: $d\mathbf{y}/dt = A \mathbf{y}$. If the origin is stable with respect to SYSTEM2, then the origin is also stable with respect to SYSTEM1.

(10) Consider $d^2y/dt^2 + p(t)y = q(t)$. Suppose $p$ has a power-series expansion about $t = 0$ with radius of convergence 2 and $q$ has a power-series expansion with radius of convergence 1.

(a) What can you say about the radius of convergence of the power-series expansion of $y$ in powers of $t$?

(b) What about the series in powers of $t - 3$?

(c) Suppose $y(0) = 1$ and $y'(0) = 3$. Find, in terms of $p$ and $q$, the coefficient of $t^2$ in this power-series expansion.

(d) Same question as (a) but now assume $q(t) \equiv 0$.

(11) Find a formula for the phase curves (curves in the phase plane) of the system $x' = -2y$, $y' = 3x$.

(12) Consider the system $\mathbf{x}' = A \mathbf{x}$, where the $3 \times 3$ matrix $A$ has a real eigenvalue $-2$ and a pair of complex conjugate eigenvalues. Prove there exists a solution $\mathbf{x}(t)$ ($t \geq 0$) which traces out a straight line segment in three-dimensional space.

(13) Find general solution to $u'' - u = 2e^{4t}$.

(14) Find general solution to $u'' + u = \sin(\mu t)$, where $\mu$ is a constant.

(15) (Continuation: Extra credit) Do the case $\mu = 1$. 

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(16) Find a continuously differentiable particular solution to
\[ \frac{d^2 u}{dt^2} + u = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \]

(17) Solve \( y' + 2xy = p(x), \ y(0) = 1 \), first for \( p(x) = x \), then for \( p(x) = x^2 \).

(18) Find general solution of \( tdx/dt = x \).

(19) Explain why phase curves of an autonomous \( 2 \times 2 \) system can’t cross, whereas those of a nonautonomous system can. Can phase curves of a \( 3 \times 3 \) autonomous system cross?

(20) Predator-prey system: \( dx/dt = x - xy, \ dy/dt = -2y + xy \). Find approximate formula for the period of the cycles very near the equilibrium point \((2, 1)\) in the \( x-y \) plane.

(21) (Continuation: Extra credit.) Prove: If \( C \) is a cycle then the line integral
\[ \int_{C} \frac{dy}{xy - 2y} \]
is equal to the period of the cycle.

(22) A basin contains 100 gallons of pure water. At time zero, a saline solution (containing 2 grams of salt per gallon) is introduced to the basin at the rate of 5 gallons per minute. The well-mixed solution is pumped out at the same rate. What is the concentration of salt (in grams per gallon) of the solution in the basin \( t \) minutes later?

(23) Find two solutions to \( y' = \sqrt{|y|}, \ y(0) = 0 \). In the general case \( y' = f(y) \), what aspect of \( f \) is related to this failure of uniqueness? Extra credit: Find a 3rd solution.

(24) Write down the Picard iteration for \( dy/dx = xy^2, \ y(0) = 1 \). (Calculate first few iterates.)

(25) (“Time must have a stop.”) Give example of an equation \( dx/dt = f(x) \) for which some solutions are defined for all \( t \) and some are not.

(26) Find by one step of Euler’s method an approximation to \( x(1) \), where \( x' = x - 3y \) and \( y' = x + 2y, \ x(0) = y(0) = 1 \).

(27) Estimate the rate of temperature decay (explain what is meant by ‘rate’) for large time in a heat-conducting rod of length 4, if the temperature satisfies the heat equation \( u_t = 2u_{xx} \), with the boundaries maintained at zero temperature. Are there exceptional solutions with faster decay?

(28) Find first 5 terms of the power series (about \( t = 0 \)) of the solution to \( y'' + \cos(t)y = 1, \ y(0) = 1, y'(0) = 2 \). For what \( t \) does this series converge?

(29) If \( A \) is a real symmetric \( 3 \times 3 \) matrix with eigenvalues \(-1, -3, -5\), and \( x(t) \) is a solution to \( dx/dt = Ax \), describe the behavior of \( x(t) \) for large \( t > 0 \).