INSTRUCTIONS: You will be allowed 4 hours for this exam. It is closed book, closed notes, and you may not use a calculator. Do the problems you know first. Indicate your answer clearly and show all your work. No work, no credit.
Problem 1. Calculate the integral of \( f(x, y) = xy + \frac{3}{x} \) over the rectangle \([1, 2] \times [-1, 1]\).
Problem 2. Find the volume of the parallelepiped determined by the vectors $a = \langle 1, 2, 3 \rangle$, $b = \langle -1, 1, 2 \rangle$ and $c = \langle 2, 1, 4 \rangle$. 
Problem 3. Justify that the limit does not exist.

\[ \lim_{(x,y) \to (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}. \]
Problem 4. If \( \mathbf{a} \cdot \mathbf{b} = \sqrt{3} \) and \( \mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle \), find the angle between \( \mathbf{a} \) and \( \mathbf{b} \).
**Problem 5.** Use the change of variables

\[ u = y + x \]
\[ v = y - x \]

to integrate the function \( f(x, y) = y^2 - x^2 \) over the region in the \( xy \)-plane bounded by the four lines

\[ y - x = 1, \ y + x = -1 \]
\[ y - x = -1, \ y + x = 1. \]
Problem 6. Find the scalar and vector projections of $\mathbf{b} = (5, -1, 4)$ onto $\mathbf{a} = (-2, 3, -6)$. 
**Problem 7.** Determine whether the lines $L_1$ and $L_2$ are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$L_1: \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 1}{-3}$

$L_2: \frac{x - 3}{1} = \frac{y + 4}{3} = \frac{z - 2}{-7}$
Problem 8. Use Green’s Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x,y) = \left( \sqrt{x^2 + 1}, \arctan x \right) \) and \( C \) is the triangle from \((0,0)\) to \((1,1)\) to \((0,1)\) to \((0,0)\).
Problem 9. Find the total length of the parametrized curve

\[ x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq \frac{\pi}{2}. \]
Problem 10. Find parametric equations for the tangent line to the curve with the parametric equations

\[ x = e^t, \quad y = te^t, \quad z = te^{t^2}; \]

at the specified point \((1, 0, 0)\).
Problem 11. Evaluate the line integral

\[ \int_C z^2 \, dx + x^2 \, dy + y^2 \, dz \]

where the curve \( C \) is the line segment from \((1, 0, 0)\) to \((4, 1, 2)\).
Problem 12. Given the vector field

\[ \mathbf{F}(x, y) = \left< \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \right>. \]

(a) Show that \( \mathbf{F} \) is conservative.
(b) Find a potential function \( f(x, y) \) such that \( \mathbf{F} = \nabla f \).
Problem 13. Let \( w = w(u, v) \) and \( u = y/x \) and \( v = x^2 + y^2 \).

(a) Express \( xw_x + yw_y \) in terms of \( w_u, w_v \) and \( u, v \).

(b) Find \( xw_x + yw_y \) in case \( w = v^5 \).
Problem 14. Find a set of parametric equations for the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]
Problem 15. Let $T$ be the tetrahedron bounded by the $xy$-plane, the $xz$-plane, the $yz$-plane, and the plane given by

$$4x + y + 2z = 4.$$ 

Set up the triple integral to compute the volume of $T$ by integrating in $z$ last. Do not evaluate the integrals you set up!
Problem 16. Use cylindrical coordinates to integrate $f(x, y, z) = 2z$ over the solid bounded above by $z = 1 - 4x^2 - 4y^2$ and below by $z = 3\sqrt{x^2 + y^2} - 6$. 
Problem 17. Use spherical coordinates to integrate the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the solid region, in $\mathbb{R}^3$, bounded by the $xz$-plane, the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{3(x^2 + y^2)}$. 
Problem 18. Let $f(x, y) = x^2y^2 - x$.
(a) Write the equation of the tangent plane to the graph of $f$ at $(2, 1, 2)$.
(b) Find the directional derivative of $f$ at $(2, 1)$ in the direction $-i + j$. 
Problem 19.
(a) Write the Lagrange Multiplier equations for the point(s) on the surface
\[ x^4 + y^4 + z^4 + xy + yz + zx = 6 \]
at which \( x \) is largest. Do not solve the equations.
(b) If \( x \) is largest at the point \((x_0, y_0, z_0)\), determine the equation of the plane tangent to the surface at that point.
Problem 20. Let \( x^2 + y^3 - z^4 = 1 \) and \( z^3 + z x + x y = 3 \) be two surfaces. These surfaces intersect on a curve passing through the point \((x, y, z) = (1, 1, 1)\). Determine the value of \( \frac{dy}{dx} \) at this point \((1, 1, 1)\) along the curve. *Hint: use total differentials.*