Math-1220-01 Final Exam

11 December 2012

NAME:_____________________________________

Some of the problems on this exam instruct you to select 2 of 3 integrals or series to evaluate. You must clearly indicate which 2 you have chosen to solve. If you work on all 3, indicate which 2 you wish to submit for a grade.

Write neatly and show all work.
Evaluate 2 of the following 3 integrals. In the case of divergent improper integrals, demonstrate that the integral is divergent.

\[
\begin{align*}
(1) & \quad \int x \cos(2x) \, dx \\
(2) & \quad \int x^2 e^x \, dx \\
(3) & \quad \int \sin(\ln x) \, dx
\end{align*}
\]
Evaluate 2 of the following 3 integrals. In the case of divergent improper integrals, demonstrate that the integral is divergent.

(1) \[ \int \frac{x^5}{\sqrt{x^2 + 2}} \, dx \]  
(2) \[ \int \frac{x^2}{(3 + 4x + 4x^2)^{3/2}} \, dx \]  
(3) \[ \int \frac{x^2 + 2x - 1}{x^3 - x} \, dx \]
Evaluate 2 of the following 3 integrals. In the case of divergent improper integrals, demonstrate that the integral is divergent.

\[ (1) \int_{-1}^{1} \frac{1}{x^2 - 2x} \, dx \quad (2) \int_{2}^{\infty} \frac{1}{x^2 + 2x - 3} \, dx \quad (3) \int_{1}^{\infty} \frac{\ln x}{x^3} \, dx \]
Approximate \( \int_{3}^{6} \ln(x^2 + 4) \, dx \) with \( n = 6 \) using (i) the midpoint rule, (ii) the trapezoid rule, and (iii) Simpson’s rule.
Sketch 3 diagrams illustrating the geometry behind the approximations given by (i) the mid-point rule, (ii) the trapezoid rule, and (iii) Simpson’s rule.
Suppose $y = e^{5x}$ satisfies the differential equation $y'' - y' + Cy = 0$. Find $C$. 
Suppose a population is modelled by the differential equation \( \frac{dP}{dt} = 0.8P \left( 1 - \frac{P}{350} \right) \).

(a) What is the name given to this type of population model?

(b) For what values of \( P \) is the population increasing?

(c) If \( P(0) = 15 \), what is \( \lim_{t \to \infty} P(t) \)?
A tank contains 500 L of brine with 10 kg of dissolved salt. Pure water enters the tank at 7 L/min. The solution is kept thoroughly mixed and drains from the tank at 7 L/min. How much salt is in the tank after 15 minutes?
Solve the differential equation $x \ln(x)y' + y = xe^x$
Use Euler's method with $h = .25$ to approximate $y(1)$, where $y(x)$ is the solution to the initial value problem

$$y' = xy - x^2$$
$$y(0) = 1$$
For 2 of the 3 following series, either find the sum or show that the series is divergent.

\((1)\) \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\)  
\((2)\) \(\sum_{n=1}^{\infty} \frac{7^{n+1}}{8^n}\)  
\((3)\) \(\sum_{n=1}^{\infty} \left[\cos \left(\frac{1}{n^2}\right) - \cos \left(\frac{1}{(n+1)^2}\right)\right]\)
For 2 of the 3 following series, state whether the series is absolutely convergent, conditionally convergent, or divergent.

(1) \( \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1} \)  
(2) \( \sum_{n=1}^{\infty} \frac{\sin(2n)}{1 + 3^n} \)  
(3) \( \sum_{n=1}^{\infty} \left( \frac{-2n}{3n + 1} \right)^6 \)
For each of the following statements, indicate whether the statement is true or false.

1. If $a_n \geq 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ is conditionally convergent.

2. If $\sum b_n$ is absolutely convergent, then $\sum (-1)^n b_n$ is convergent.

3. If $c_n \geq 0$ and $\sum c_n$ converges, then $\sum (-1)^n c_n$ converges.

4. If $d_n \geq 0$ and $\sum (-1)^n d_n$ is conditionally convergent, then $\sum d_n$ is divergent.
Find a power series representation of $f(x) = \frac{3}{1+2x^2}$. Determine the interval of convergence.
Find the radius and interval of convergence for the power series \( \sum_{n=1}^{\infty} \frac{n}{4^n} (x + 1)^n \)
Find the Taylor series expansion for \( f(x) = \frac{1}{x} \) centered at \( a = 2 \).
Suppose \( a_n = \frac{\sin n}{1 + \sqrt[3]{n}} \). Find \( \lim_{n \to \infty} a_n \).
Give the Cartesian coordinates of the point whose polar coordinates are \( \left( \sqrt{2}, \frac{5\pi}{4} \right) \)
Eliminate the parameter to find a Cartesian equation of the curve described by

\[
\begin{align*}
x &= 1 - t^2 \\
y &= t - 2
\end{align*}
\]
Extra credit. Return to problems 1, 2, 3, 11, and 12. Evaluate some or all of the 3 integrals and 2 series you did not evaluate earlier.