Instructions

This exam has two parts.
**Part. 1** consists of short-answer questions; write only your answers. Each question is worth 5 points.
**Part. 2** is a set of workout problems. Show all your work in the space provided. Each question here is worth 10 points.
Calculators are allowed, but not scientific ones; models equivalent to or lower than TI-84 are allowed.
You have 4h to answer the questions. Take your time, read the whole text before starting.
There is a total of 9 pages and 200 points.

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Formulas

\[
\begin{align*}
\cos^2(x) + \sin^2(x) &= 1 \\
\tan^2(x) + 1 &= \sec^2(x) \\
\sin(2x) &= 2 \sin(x) \cos(x) \\
\cos(2x) &= 1 - 2 \sin^2(x) = \cos^2(x) - \sin^2(x) \\
\sqrt{a^2 - b^2x^2} &\Rightarrow x = \frac{a}{b} \sin \theta \\
\sqrt{b^2x^2 - a^2} &\Rightarrow x = \frac{a}{b} \sec \theta \\
\sqrt{a^2 + b^2x^2} &\Rightarrow x = \frac{a}{b} \tan \theta \\
\int \sec(x) \, dx &= \ln |\sec(x) + \tan(x)| + C \\
\int \tan(x) \, dx &= \ln |\sec(x)| + C \\
\int \frac{du}{a^2 + u^2} &= \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C \\
\int \frac{dx}{\sqrt{1 - x^2}} &= \arcsin(x) + C
\end{align*}
\]

Midpoint approx.:

\[
\int_a^b f(x) \, dx \approx \Delta x \left[ f(x_1) + f(x_2) + \ldots + f(x_n) \right]
\]

\[
x_i = \frac{x_{i-1} - x_{i-2}}{2}
\]

Trapezoidal approx.:

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]
\]

Simpson approx.:

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n) \right]
\]

Arc Length: \( L = \int_{x=a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

Arc Length: \( L = \int_{y=a}^{b} \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \)

Surface Area (rotating around \( x \)-axis): \( S = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

Surface Area (rotating around \( y \)-axis): \( S = \int_a^b 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \)
1. (100 points) **Short-Answer: Write only answers**

1. (5 pts) Give an upper bound of the sequence \( \{e^{\frac{2}{n^2}}\}_{n=5}^\infty \).

2. (5 pts) Write the differential equation \( y' + y = x - xy + 1 \) as a separable differential equation of the form \( f(y)dy = g(x)dx \).

3. (5 pts) Setup, but do **not evaluate** an integral for the arc length of the curve \( \ln(x) = \frac{y}{3} \) between \((1, 0)\) and \((e, 3)\).

4. (5 pts) Does the integral \( \int_1^3 \ln(x - 1)dx \) converge or diverge?

5. (5 pts) If \( \sum_{n=0}^{+\infty} \cos(x)^n = 2 \) then find \( x \).

6. (5 pts) Convert \( \int \frac{x}{\sqrt{25 + x^2}}dx \) to a trigonometric integral, **without evaluating it**.

7. (5 pts) Evaluate the indefinite integral \( \int \frac{\arctan(x)}{1 + x^2}dx \).

8. (5 pts) Does the series \( \sum_{n=5}^{+\infty} \frac{n^2 - 3n + 2}{(n - 1)(n + 1)} \) converge or diverge?

9. (5 pts) Find the minimal \( k \) such that \( \left| \sum_{n=1}^{k} (-1)^n \frac{n}{n^2 + 1} - \sum_{n=1}^{+\infty} (-1)^n \frac{n}{n^2 + 1} \right| \leq 0.2 \).

10. (5 pts) Find the limit: \( \lim_{n \to +\infty} \frac{n^4}{n!} \).
11. (5 pts) Setup, but do not evaluate an integral for the surface area obtained by rotating the curve \( x = \arctan(y^2) \) between \((1,0)\) about the \( y \) axis for \( 1 \leq y \leq 3 \).

12. (5 pts) For which values of \( \alpha > 0 \) does the series \( \sum_{n=1}^{+\infty} \left( -\frac{1}{n^\alpha + 1} \right)^n \) converge?

13. (5 pts) Use Simpson’s rule with \( n = 4 \) to evaluate \( \int_{0}^{1} e^{-x^2} \, dx \).

14. (5 pts) Find the sum of the series: \( 1 + \frac{\ln(3)}{1!} + \frac{(\ln(3))^2}{2!} + \frac{(\ln(3))^3}{3!} + \ldots \)

15. (5 pts) What is the radius of convergence of the power series \( \sum_{n=0}^{+\infty} \sqrt{n} x^n \)

16. (5 pts) Evaluate the indefinite integral \( \int \tan(x) \sec^2(x) \, dx \)

17. (5 pts) Evaluate the integral \( \int_{1}^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \)

18. (5 pts) Use Euler’s method with a step \( h = 0.1 \) to estimate the value of \( y(1.5) \) where \( y \) the the solution of the initial-value problem: \( y' = x \sin(y) \), \( y(1) = 0 \).

19. (5 pts) If \( f(x) = (x - 1)e^x \) satisfies the differential equation \( y' - y = Ce^x \), then find \( C \)

20. (5 pts) Find the following limit: \( \lim_{x \to 0} \frac{\sin(x)}{\ln(x)} \)
2. (100 points) **Workout Problems: Show Work**

1. (10 pts) Evaluate the following integral: \[ \int_0^{\pi/4} \sin^4(x) \cos^5(x) dx \]

2. (10 pts) In a tank of pure water of volume \( V = 5m^2 \), a mixture containing 20\% of crude oil spills from the top at a rate \( r = 1m^2/s \). The solution is kept thoroughly mixed, and is drained from the bottom at the same rate \( r = 1m^2/s \). Find the concentration of oil in function of \( t \). What happens when \( t \to +\infty \)?
3. (10 pts) Use the integral test to prove that the following series converge: \( \sum_{n=1}^{+\infty} n^2e^{-n} \).

4. (10 pts) Use the first 3 terms of the McLaurin expansion of \( e^{-x^2} \) to compute an approximation of \( \int_{0}^{1} e^{-x^2} \, dx \).
5. (10 pts) Evaluate the following integral: \[ \int_0^3 \frac{x^3}{\sqrt{3 + x^4}} \, dx \]

6. (10 pts) Solve the following differential equation: \[ 2 \frac{dy}{dx} = \frac{1 + y^2}{1 - x^2} \]
7. (10 pts) Find the arch length of the curve: $y = \ln(\sec(x))$ from $x = 0$ to $x = \frac{\pi}{4}$.

8. (10 pts) Find a power series representation for $\frac{1}{x+3}$. What is its radius?
9. (10 pts) Prove that the power series \(-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\) converges for \(x = -1\). Use this fact to find the value of the series: \(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\)

10. (10 pts) Evaluate the following improper integral: \(\int_{0}^{1} x \ln(x) dx\)