Math 1310 Final Exam

December 11, 2014

NAME:__________________________________________

INSTRUCTOR:____________________________________

Write neatly and show all your work in the space provided below each question. You may use the back of the exam pages if you need additional space to show your work. **Answers without work and illegible answers will not receive credit.**

Each student is expected to follow Tulane’s Code of Academic Conduct. Any violation will be reported to the Honor Board for investigation.

A scientific calculator from the Mathematics Department’s list of approved calculators is permitted on the exam. A calculator capable of symbolic manipulation or equipped with a computer algebra package is not permitted.
Part I - Short answer (3 points each – 60 points total)

1. Evaluate the limit \( \lim_{x \to 2} \frac{\sqrt{x+2} - 2}{x-2} \).

Answer: 

2. Find the slope of the tangent line to the curve \( ye^x + \sin(x y) = 1 \) at the point \((0, 1)\).

Answer: 

3. Find the interval(s) where the function $f(x) = xe^{-x}$ isconcave up.

Answer:

4. Determine all horizontal and vertical asymptotes of the function $f(x) = \frac{1 + e^{-x}}{1 - e^{-x}}$.

Answer:
5. Evaluate the indefinite integral \( \int \frac{x^3}{1+x^4} \, dx \).

Answer:

6. Find the volume of the solid obtained by rotating the region bounded by the curve \( y = 4 - x^2 \), the \( x \)-axis and the \( y \)-axis about the line \( y = 4 \).

Answer:
7. Evaluate the indefinite integral \( \int x \sin(2x) \, dx \).

Answer:

8. Evaluate the indefinite integral \( \int \sec^3 t \tan t \, dt \).

Answer:
9. Evaluate the indefinite integral \( \int \frac{2x^3 + 3}{x(x+1)} \, dx \).

Answer:

10. Determine whether the improper integral \( \int_0^\infty e^{-x/3} \, dx \) converges or diverges. If the integral converges, find its value.

Answer:
11. Write down an integral which computes the arc length of the curve \( y = \tan x \) between the points \((0, 0)\) and \((\pi/4, 1)\). Do not attempt to evaluate the integral.

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\text{Answer:}
\]

12. Find the general solution of the differential equation \( \frac{dy}{dx} = xy^2 \).

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\text{Answer:}
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13. Find the particular solution of the initial value problem \( \frac{dy}{dx} + 4xy = x, y(0) = 1. \)

Answer:

14. Find the slope of the tangent line to the parametric curve \( x = (t + 1)^2, y = t^3 - 4t \) at the point \((2, 0)\).

Answer:
15. Find the area of the region that is inside both the curves \( r = 2 \cos \theta \) and \( r = 2 \sin \theta \).

Answer:

16. Determine the limit of the sequence \( \lim_{n \to \infty} \frac{n^2 + n - 1}{2n^2 - n + 3} \)

Answer:
17. Find the value of the geometric series \( \frac{8}{3} - \frac{16}{9} + \frac{32}{27} - \frac{64}{81} + \cdots \).

Answer:

18. Determine if the series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \) converges absolutely, converges conditionally, or diverges.

Answer:
19. Determine the radius of convergence of the power series \( \sum_{n=1}^{\infty} \frac{2^n n^2}{n!} x^n \).

Answer:

20. Evaluate \( \lim_{x \to 0} \frac{\cos(3x^2) - 1}{x^4} \).

Answer:
Part II - Long answer (8 points each - 40 points total)

1. A box with a square base and open top is to be built.
   
   (a) (2 points) In terms of the length \( \ell \) of the box’s base and the height \( h \) of the box, write down formulas for the volume \( V \) of the box and the surface area \( A \) of the box.

   Answer: 

   \( V = \ell^2 h \)
   \( A = 4\ell^2 + 2\ell h \)

   (b) (6 points) If the box is required to have a volume of exactly 500 cubic inches, find the dimensions of the box that will minimize the surface area of the box.

   Answer: 

   \( \ell = \sqrt[3]{\frac{500}{h}} \)
   \( A = 4\left(\sqrt[3]{\frac{500}{h}}\right)^2 + 2\sqrt[3]{\frac{500}{h}} h \)

   For minimum surface area, take derivative and solve for \( h \).
2. This problem will evaluate the integral $\int_{0}^{1.5} x^2 \sqrt{9 - x^2} \, dx$ in several steps.

(a) (3 points) Using an appropriate trigonometric substitution, convert $\int_{0}^{1.5} x^2 \sqrt{9 - x^2} \, dx$ into a trigonometric integral. (Be sure that you change your limits of integration appropriately.)

Answer:

(b) (5 points) Using the identities $\cos^2(\theta) = \frac{1}{2} (1 + \cos(2\theta))$ and $\sin^2(\theta) = \frac{1}{2} (1 - \cos(2\theta))$, evaluate the trigonometric integral.

Answer:
3. The population $P(t)$ of wolves in a certain ecosystem as a function of time $t$ (measured in years) satisfies the following differential equation,

$$\frac{dP}{dt} = 5 - 0.1P$$

(a) (4 points) Determine the general solution for $P(t)$.

Answer:

(b) (2 points) Initially (at time $t = 0$), there are 100 wolves. Use this information to find the particular solution for $P(t)$.

Answer:

(c) (2 points) Using the particular solution from part (b), determine the size of the wolf population as $t \to \infty$.

Answer:
4. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n+1}}$.

(a) (3 points) Use the Ratio Test to determine the radius of convergence of the power series.

Answer:

(b) (5 points) Determine the interval of convergence of the power series. For each endpoint, justify whether the series converges or diverges by citing an appropriate test.

Answer:
5. Recall that the Taylor series (centered at $a = 0$) for the exponential function is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$ 

(a) (2 points) Find the Taylor series (centered at $a = 0$) for $e^{-x^2}$.

Answer:

(b) (4 points) Find the Taylor series (centered at $a = 0$) for

$$\int_0^x e^{-t^2} \, dt$$

and express your answer using summation notation (i.e. similar to how the Taylor series for $e^x$ is presented in this problem).

Answer:

(b) (2 points) Use the first four non-zero terms of an appropriate Taylor series to approximate

$$\int_0^{0.5} e^{-x^2} \, dx.$$ 

Give your answer to six decimal places.

Answer: