## ANSWER SHEET  MATH 2240 FINAL EXAM

Name: 

Instructor: 

1. Do NOT SEPARATE answer sheet from rest of exam. 
2. CIRCLE the answer to each problem INSIDE this exam. 
3. Circle your answer a SECOND TIME on this page. 

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<tr>
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This exam contains 28 problems. The first 20 are multiple choice and are 9 points each. Problems 21–28 are true or false and are 5 points each. A small table of Laplace transforms is provided at the last page of the exam. You may detach it from the rest of the exam.

Record the answers on the cover sheet. Clearly mark one answer only. If two or more answer are marked, no credit will be given. No partial credit will be given if a wrong answer is marked.

NOTE: Even though the total number of points is 220, your grade will be calculated out of 200 points, which means that the highest possible grade is 200 out of 200.

Calculators are allowed. Good luck!

1. The inverse Laplace transform of $\frac{7s^2 + 3s + 16}{(s + 1)(s^2 + 4)}$ is:

   (a) $4e^t + 3\sin(2t)$ (b) $4e^{-t} - 3\sin(2t)$ (c) $4e^{-t} + 3\cos(2t)$ (d) $3e^t - 4\cos(2t)$

   (e) none of the above

2. Let $x(t)$ be a solution of the following initial value problem:

   \[
   \begin{cases}
   x'' + 3x' + 2x = 3e^{-t}, \\
   x(0) = 1, \ x'(0) = 0.
   \end{cases}
   \]

   Then $x'(1)$ is equal to:

   (a) $3e^{-1} - \frac{5}{2}e^{-2} + \frac{1}{2}e^{-4}$ (b) $-3e^{-1} + 5e^{-2}$ (c) $-3e^{-1} + 5e^{-2} - 2e^{-4}$ (d) $-2e^{-4}$

   (e) none of the above

3. A first-order system equivalent to the third-order equation $x''' + x'x'' = \sin t$ is:

   (a) \[
   \begin{cases}
   u'_1 = u_2 \\
   u'_2 = -u_1u_2 + \sin t
   \end{cases}
   \]

   (b) \[
   \begin{cases}
   u'_1 = u_2 \\
   u'_2 = u_3 \\
   u'_3 = -u_2u_3 + \sin t
   \end{cases}
   \]

   (c) \[
   \begin{cases}
   u_1 = x' \\
   u_2 = x'' \\
   u'_2 = -u_1u_2 + \sin t
   \end{cases}
   \]

   (d) this equations does not contain $x$ and thus cannot be written as a system of equations

   (e) none of the above
4. Consider the system \[
\begin{align*}
    x' &= x(6 - 3x) - 2xy, \\
    y' &= y(5 - y) - xy.
\end{align*}
\] The equilibrium point which is located in the interior of the second quadrant of the phase plane is:

    (a) nodal source  (b) saddle point  (c) nodal sink  (d) spiral source  
    (e) none of the above

5. The undamped system \[
\begin{align*}
    \frac{2}{5} x'' + kx &= 0, \\
    x(0) &= 2, \quad x'(0) = v_0
\end{align*}
\] is observed to have period \(\pi/2\) and amplitude 2. Then \(k\) and \(v_0\) are equal to:

    (a) \(k = 4, \ v_0 = 0\)  (b) \(k = -\frac{32}{5}, \ v_0 = 4\)  (c) \(k = \frac{2}{5}, \ v_0 = 0\)  (d) \(k = \frac{32}{5}, \ v_0 = 0\)  
    (e) none of the above

6. Consider the planar \((2 \times 2)\) system \[
\begin{align*}
    x' &= -7x + 10y, \\
    y' &= -5x + 8y.
\end{align*}
\] Then the origin \((0, 0)\) in the phase plane is:

    (a) a spiral source  (b) a nodal sink  (c) a nodal source  (d) a spiral sink  
    (e) none of the above

7. Suppose the linear planar \((2 \times 2)\) system \(y' = Ay\) has two solutions \[
    y_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{and} \quad y_2(t) = e^{3t} \begin{pmatrix} 4 \\ -2 \end{pmatrix}.
\] Which of the following is NOT a solution of \(y' = Ay\)?

    (a) \(e^{-2t} \begin{pmatrix} -2 \\ -8 \end{pmatrix}\)  (b) \(e^{3t} \begin{pmatrix} -4 \\ 2 \end{pmatrix}\)  (c) \(e^{-2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - e^{3t} \begin{pmatrix} 4 \\ -2 \end{pmatrix}\)  
    (d) \(4e^{-2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 7e^{3t} \begin{pmatrix} 4 \\ -2 \end{pmatrix}\)  
    (e) none of the above
8. Let \( x(t) \) be a solution of the following initial value problem:
\[
\begin{cases}
x' = \frac{2tx}{1+x}, \\
x(0) = 1.
\end{cases}
\]
Then \( x(\sqrt{e}) \) is equal to:

(a) \( e + 1 \)  
(b) \( e \)  
(c) \( \sqrt{e} \)  
(d) \( e^2 \)  
(e) none of the above

9. Let \( y(t) \) be the solution of the system
\[
y' = \begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]
subject to the initial condition \( y_1(0) = 2013, y_2(0) = 2014 \). Which of the following is true?

(a) \( y(t) \) spirals towards the origin as \( t \to \infty \), moving clockwise
(b) \( y(t) \) spirals towards the origin as \( t \to \infty \), moving counterclockwise
(c) \( y(t) \) converges to the origin as \( t \to \infty \), tangent to some line
(d) \( y(t), t \geq 0 \), is unbounded  
(e) none of the above

10. The Laplace transform of \( e^{-t}(t^2 + 3t + 4) \) is:

(a) \( -\frac{2 + 3s + 4s^2}{s^3} \)  
(b) \( \frac{2 + 3s + 4s^2}{s^3} e^{-s} \)  
(c) \( \frac{9 + 11s + 4s^2}{(s + 1)^3} e^{-s} \)  
(d) \( \frac{9 + 11s + 4s^2}{s^3 + 3s^2 + 3s + 1} \)

(e) none of the above

11. Let \( y(t) \) be the solution of the system
\[
y' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]
subject to the initial condition \( y_1(0) = 5, y_2(0) = 1 \). Then \( y(\pi) \) satisfies:

(a) in the I quadrant \( (y_1(\pi) > 0, y_2(\pi) > 0) \)
(b) in the II quadrant \( (y_1(\pi) < 0, y_2(\pi) > 0) \)
(c) in the III quadrant \( (y_1(\pi) < 0, y_2(\pi) < 0) \)
(d) in the IV quadrant \( (y_1(\pi) > 0, y_2(\pi) < 0) \)  
(e) none of the above
12. The Laplace transform of

\[ f(t) = \begin{cases} 
  t, & 0 \leq t \leq 1, \\
  1, & 1 < t < \infty 
\end{cases} \]

is:

(a) \( -\frac{e^{-s}}{s^2} + \frac{1}{s^2} \)  
(b) \( \frac{1 - e^{-s}}{s} \)  
(c) \( \frac{1 + e^{-s}}{s^2} \)  
(d) undefined  
(e) none of the above

13. Consider the Lorenz system

\[ \begin{align*}
  x' &= -x + y, \\
  y' &= rx - y - xz, \\
  z' &= -z + xy,
\end{align*} \]

where \( r \) is a positive constant. The origin is an asymptotically stable equilibrium point of this system if:

(a) \( r > 1 \)  
(b) \( 1 < r < 4 \)  
(c) \( r < 1 \)  
(d) \( r \neq 1 \)  
(e) none of the above

14. The interval of existence of the solution of the initial value problem

\[ \begin{align*}
  tx' + 2x &= \sin t, \\
  x(\pi/2) &= 0.
\end{align*} \]

is:

(a) the whole real line  
(b) the whole real line except for the point \( t = 0 \)  
(c) no solution exists in any interval  
(d) \( (0, \infty) \)  
(e) none of the above

15. Let \( x(t) \) be a solution of the following initial value problem:

\[ \begin{align*}
  tx' + 2x &= \sin t, \\
  x(\pi/2) &= 0.
\end{align*} \]

Then \( x(\pi) \) is equal to:

(a) \( \frac{1}{\pi} \)  
(b) \( \frac{1}{\pi^2} \)  
(c) \( -\frac{1}{\pi^2} \)  
(d) undefined  
(e) none of the above

16. A tank contains 100 gal of pure water. At time \( t = 0 \), a solution containing 2 lb of salt per gallon begins to enter the tank at the rate 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant. How much salt is in the tank after 2013 minutes?

(Assume that the solution in the tank is kept perfectly mixed at all times).

(a) \( \approx 100 \) lb  
(b) \( \approx 6039 \) lb  
(c) \( \approx 200 \) lb  
(d) \( \approx 2013 \) lb  
(e) none of the above
17. Let $x(t)$ be a solution of the following initial value problem:

$$
\begin{cases}
    x'' + 2x' + 2x = 2 \cos (2t), \\
    x(0) = -2, \quad x'(0) = 0.
\end{cases}
$$

Then $x'(\pi/2)$ is equal to:

(a) $\frac{2}{5} - \frac{9}{5} e^{-\pi/2}$  
(b) $-\frac{13}{5} e^{-\pi/2}$  
(c) $\frac{1}{5}$  
(d) $-\frac{1}{5} + \frac{9}{5} e^{-\pi/2}$  
(e) none of the above

18. Let $y(t)$ be the solution of the system

$$
y' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
$$

subject to the initial condition $y_1(0) = 2, y_2(0) = -1$. Then $y(\pi)$ is:

(a) in the I quadrant ($y_1(\pi) > 0, y_2(\pi) > 0$)  
(b) in the II quadrant ($y_1(\pi) < 0, y_2(\pi) > 0$)  
(c) in the III quadrant ($y_1(\pi) < 0, y_2(\pi) < 0$)  
(d) in the IV quadrant ($y_1(\pi) > 0, y_2(\pi) < 0$)  
(e) none of the above

19. Let $x(t)$ be the solution of the initial value problem

$$
\begin{cases}
    x' = 9x - x^3, \\
    x(0) = 1.5.
\end{cases}
$$

Then the limit $\lim_{t \to \infty} x(t)$ is equal to:

(a) 0  
(b) 3  
(c) -3  
(d) undefined  
(e) none of the above

20. Let $y(t)$ be the solution of the system

$$
y' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
$$

subject to the initial condition $y_1(0) = 1, y_2(0) = 2$. Then $y(\pi)$ is:

(a) in the I quadrant ($y_1(\pi) > 0, y_2(\pi) > 0$)  
(b) in the II quadrant ($y_1(\pi) < 0, y_2(\pi) > 0$)  
(c) in the III quadrant ($y_1(\pi) < 0, y_2(\pi) < 0$)  
(d) in the IV quadrant ($y_1(\pi) > 0, y_2(\pi) < 0$)  
(e) none of the above
Problems 21–28 are true (T) or false (F). It is NOT necessary to justify your answers.

21. The matrix \[
\begin{pmatrix}
2 - x & 0 & 0 \\
-1 & -x & 2 \\
0 & -2 & 5 - x
\end{pmatrix}
\] is nonsingular for any \( x \) such that \( |x| > 4 \).

\[(T) \quad (F)\]

22. Notice that \((x(t) = t, y(t) = e^t)\) is a solution of the system
\[
\begin{align*}
x' &= 1 - e^x (y - e^x), \\
y' &= 2e^x - y. 
\end{align*}
\]

Is the following statement true or false: The solution of the system (1) with the initial data \( x(0) = y(0) = 0 \) approaches the equilibrium point \((0, 2)\) as \( t \to \infty \)?

\[(T) \quad (F)\]

23. The vectors \( v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \ v_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} \) are linearly independent.

\[(T) \quad (F)\]

24. The system
\[
\begin{align*}
x' &= y + x \left( 3 - \sqrt{x^2 + y^2} - \frac{2}{\sqrt{x^2 + y^2}} \right), \\
y' &= -x + y \left( 3 - \sqrt{x^2 + y^2} - \frac{2}{\sqrt{x^2 + y^2}} \right),
\end{align*}
\]
has two (one attracting and one repelling) limit cycles.

*Hint:* Use polar coordinates.

\[(T) \quad (F)\]

25. It is possible to find a function \( f(t, x) \) that is continuous and has continuous partial derivatives such that the functions \( x_1(t) = t \) and \( x_2(t) = \sin t \) are both solutions of \( x' = f(t, x) \) near \( t = 0 \).

\[(T) \quad (F)\]
26. The solution of the initial value problem

\[
\begin{align*}
    y' &= 3y^{2/3} \\
    y(0) &= 0
\end{align*}
\]

is unique in the interval where it exists.

(T) \hspace{2cm} (F)

27. For the system

\[
y' = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 4 & -2 & -3 \end{pmatrix} y
\]

the equilibrium point at the origin is asymptotically stable.

(T) \hspace{2cm} (F)

28. For any square matrices of the same dimensions \(A\) and \(B\), \(AB = BA\).

(T) \hspace{2cm} (F)
A small table of Laplace transforms

\[ \mathcal{L}\{1\}(s) = \frac{1}{s} \]
\[ \mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}} \]
\[ \mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2 + a^2} \]
\[ \mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2 + a^2} \]
\[ \mathcal{L}\{e^{at}\}(s) = \frac{1}{s - a} \]
\[ \mathcal{L}\{e^{at}\sin(bt)\}(s) = \frac{b}{(s - a)^2 + b^2} \]
\[ \mathcal{L}\{e^{at}\cos(bt)\}(s) = \frac{s - a}{(s - a)^2 + b^2} \]
\[ \mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s - a)^{n+1}} \]
\[ \mathcal{L}\{H(t-c)f(t-c)\}(s) = e^{-cs}F(s), \text{ where } F(s) = \mathcal{L}\{f(t)\}(s) \]
\[ \mathcal{L}\{e^{ct}f(t)\}(s) = F(s - c) \]
\[ \mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s) \]