MATH-2240 FINAL EXAMINATION
Friday, May 4, 2012, 8:00AM-12:00NOON

Your Instructor: ________________  Your Name: ________________

1. Do not open this exam until you are told to do so.

2. This exam has 30 problems and 18 pages including this cover sheet.

3. Do not separate the pages of this exam except the formula sheet at the very end. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.

5. **Turn off all cell phones and pagers, and remove all headphones.**

6. **You must clearly indicate your final answer for each problem to receive credit.**

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1. Let $x(t)$ be the solution of the initial value problem of the autonomous equation $x' = -3x(2-x)(x+2)(x-1)$ with $x(0) = 1.5$. Find $\lim_{t \to \infty} x(t)$.

(a). 0  (b). 1  (c). 2  (d). 3  (e). None of the above

2. Let $x(t)$ be the solution of the initial value problem $3tx' - 3x = 6t^2$, $x(1) = 0$. Find the value of $x(2)$.

(a). 1  (b). 2  (c). 3  (d). 4  (e). None of the above
3. Find the general solution of the second order linear homogeneous ODE with constant coefficients \(4x'' + 4x' + x = 0\).

   (a) \(x(t) = a + bt\)  \(\quad\) (b) \(x(t) = e^{3t/2}\)  \(\quad\) (c) \(x(t) = e^{t/2}(a + bt)\)
   (d) \(x(t) = e^{-t/2}(a + bt)\)  \(\quad\) (e) None of the above

4. Let \(x(t)\) be the solution to the initial value problem \(x'' - 2x' + 10x = 0\), \(x(0) = 0\), \(x'(0) = \frac{3}{2}\). Find \(x(\pi/2)\).

   (a) \(x(\pi/2) = -e^{-\pi/2}\)  \(\quad\) (b) \(x(\pi/2) = 0\)  \(\quad\) (c) \(x(\pi/2) = \frac{1}{2}e^{\pi/2}\)
   (d) \(x(\pi/2) = -\frac{1}{2}e^{\pi/2}\)  \(\quad\) (e) None of the above
5. Consider the planar system

\[
\begin{align*}
x' &= 4x - 10y \\
y' &= 2x - 4y
\end{align*}
\]

Then the origin \((0, 0)\) in the phase plane is a

(a). saddle point   (b). nodal sink   (c). nodal source
(d). Spiral sink   (e). None of the above

6. Suppose the linear planar system \(y' = Ay\) has two solutions \(y_1(t) = e^{-3t}(2, 4)^T\) and \(y_2(t) = e^{2t}(1, -2)^T\). Which of the following is not a solution of \(y' = Ay\)?

(a). \(e^{-3t}(1, 2)^T\)   (b). \(e^{-3t}(1, 3)^T\)   (c). \(e^{-3t}(2, 4)^T - e^{2t}(1, -2)^T\)
(d). \(4e^{-3t}(2, 4)^T + 7e^{2t}(1, -2)^T\)   (e). None of the above
7. Consider the planar linear system $y' = Ay$, with $y = (y_1, y_2)^T$. If the plot of $y_1$ vs. $t$ is periodic, which of the following is the matrix $A$?

(a). $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
(b). $\begin{pmatrix} -3 & -2 \\ -2 & -2 \end{pmatrix}$
(c). $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$
(d). $\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$
(e). none of the above

8. Solve the second order initial value problem $y'' - 4y' - 5y = 2$, $y(0) = 0$, and $y'(0) = 2$.

(a). $y(t) = \frac{1}{3}(e^{5t} - e^{-t})$
(b). $y(t) = \frac{1}{3}(e^{5t} + e^{-t})$
(c). $y(t) = \frac{2}{5}(e^{5t} - 1)$
(d). $y(t) = \frac{2}{5}(e^{5t} + 1)$
(e). None of the above
9. Find the Laplace transform of the function $f(t)$ which is defined by $f(t) = 0$ for $t < 0$, $f(t) = t$ for $0 \leq t < 5$, and $f(t) = 5$ for $t \geq 5$.

(a). \( \frac{1}{s}(1 - e^{5s}) \)  
(b). \( \frac{1}{s^2}(1 - e^{-5s}) \)  
(c). \( \frac{1}{s}(1 - e^{-5s}) \)  
(d). \( \frac{1}{s^2}(1 - e^{5s}) \)  
(e). None of the above

10. Let $y(t)$ be the solution of the initial value problem $y' = (1 + y)e^t, y(0) = 0$. Find the value of $y(ln 2)$.

(a). $e - 2$  
(b). $e - 1$  
(c). $e$  
(d). $e + 1$  
(e). none of the above
11. Let \( x(t) \) be the solution of the initial value problem \( x' - x \sin t - 2te^{-\cos t} = 0, \)
\( x(0) = 1 \). Find the value of \( x(\pi/2) \).

(a). \( e - \frac{\pi^2}{4} \)  (b). \( e \)  (c). \( e + \frac{\pi^2}{4} \)  (d). \( e - \frac{\pi^2}{2} \)  (e). none of the above

12. A biologist prepares a culture. After 1 day of growth, the biologist counts 1000 cells. After 2 days of growth, he counts 3000. Assuming a Malthusian growth model, how many cells were present initially?

(a). 100  (b). \( \frac{3000}{\ln 3} \)  (c). \( \frac{1000}{\ln 3} \)  (d). \( \frac{1000}{3} \)
(e). None of the above
13. A 1-kg mass is attached to a spring with \( k = 4 \text{kg/s}^2 \) and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies an external driving force \( f(t) = 6 \sin t \) Newtons. The system is started from equilibrium with the mass having no initial displacement nor velocity. Ignore any damping forces. Find the position of the mass as a function of time.

(a) \( 2 \sin t + \sin 2t \)  
(b) \( 2 \sin t - \sin 2t \)  
(c) \( \sin 2t - 2 \sin t \)  
(d) \( -\sin 2t - 2 \sin t \)  
(e) None of the above

14. Let \( x(t) \) be the solution of the initial value problem \( x' = x - x^2 \) with \( x(0) = 0.5 \). Find the value of \( x(1) \).

(a) \( \frac{e}{e+1} \)  
(b) \( \frac{-e}{e+1} \)  
(c) \( \frac{e}{e-1} \)  
(d) \( \frac{e}{1-e} \)  
(e) None of the above
15. Find the Laplace transform for \( f(t) = e^{\omega t} + e^{-\omega t} \), where \( \omega > 0 \) is a positive constant.

(a). \( \frac{2s}{s^2 - \omega^2} \)  
(b). \( \frac{2\omega}{s^2 - \omega^2} \)  
(c). \( \frac{2s}{s^2 + \omega^2} \)  
(d). \( \frac{2\omega}{s^2 + \omega^2} \)  
(e). none of the above

16. Find the Laplace transform of \( y(t) = e^{-t}(t^2 + 3t + 4) \).

(a). \( \frac{(2 + 3s + 4s^2)}{s^3} \)  
(b). \( \frac{(4s^2 + 11s + 9)}{(s + 1)^3} \)  
(c). \( \frac{(4s^2 + 11s + 9)}{(s - 1)^3} \)  
(d). \( \frac{(-4s^2 + 11s + 9)}{(s + 1)^3} \)  
(e). none of the above
17. Find the inverse Laplace transform of the function \( Y(s) = \frac{2s-2}{(s-4)(s+2)} \).

(a) \( e^{4t} + e^{-2t} \)
(b) \( 2e^{4t} + e^{-2t} \)
(c) \( e^{4t} + 2e^{-2t} \)
(d) \( -e^{4t} + e^{-2t} \)
(e) none of the above

18. Find the inverse Laplace transform for the function \( Y(s) = \frac{3s+1}{s^2+4s+29} \).

(a) \( e^{2t}(3 \cos 5t + \sin 5t) \)
(b) \( e^{2t}(3 \cos 5t - \sin 5t) \)
(c) \( e^{-2t}(3 \cos 5t + \sin 5t) \)
(d) \( e^{-2t}(3 \cos 5t - \sin 5t) \)
(e) none of the above
19. Solve the linear system $Ax = b$, where

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & -2 \\ -2 & 0 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

(a). $(7, 0, 1)^T$ (b). $(2, 2, 4)^T$ (c). $(0, 0, 0)^T$ (d). $(-3, -2, 1)^T$
(e). none of the above

20. The autonomous equation $x' = 5x(2 - x)^2(x + 2)(x - 1)$ has exactly four equilibrium points at $x = -2, x = 0, x = 1$, and $x = 2$. Classify each equilibrium point as either unstable or asymptotically stable. Which of the following statements is correct.

(a). All of $x = -2, x = 1, x = 2$ are unstable
(b). All of $x = -2, x = 0, x = 2$ are unstable
(c). Both $x = 0$ and $x = 2$ are asymptotically stable
(d). Both $x = 0$ and $x = 2$ are unstable (e). none of the above
For problems #21 and #22, the matrix $A$ is given by

$$A = \begin{pmatrix} 1 & 4 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 3 & 12 & 3 & 3 \end{pmatrix}$$

21. The general solution $\mathbf{x}$ for the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is
   (a). $(v + 4u, 2v, u, v)^T$    (b). $(v - 4u, -2v, u, v)^T$
   (c). $(v + 4u, u, -2v, v)^T$    (d). $(v - 4u, u, -2v, v)^T$
   (e). none of the above

22. Find a basis for the nullspace of the matrix $A$.
   (a). $(-4, 1, 0, 0)^T$, $(1, 0, -2, 1)^T$    (b). $(1, 2, 0, 1)^T$, $(4, 0, 1, 0)^T$
   (c). $(1, 0, -2, 1)^T$, $(4, 0, 5, 0)^T$    (d). $(1, -2, 0, 1)^T$, $(-4, 5, 0, 0)^T$
   (e). none of the above
23. Let $B$ be a square matrix defined by $B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & x \end{pmatrix}$. Find the value of $x$ for which the matrix $B$ has a nontrivial nullspace.

(a). $-1$  (b). 0  (c). 1  (d). 2  (e). none of the above

24. If $\vec{y}_1(t) = (e^{-4t}, e^{-4t})^T$ and $\vec{y}_2(t) = (e^{-2t}, 0)^T$ form a fundamental set of solutions to the linear homogeneous planar system $\vec{y}' = A\vec{y}$, and $\vec{y}(t)$ is a solution to $\vec{y}' = A\vec{y}$ with initial condition $\vec{y}(0) = (-1, 1)^T$, find $\vec{y}'(0)$.

(a). $(-4, 0)^T$  (b). $(0, -4)^T$  (c). $(-2, 0)^T$  (d). $(0, -2)^T$  (e). none of the above
For problems #25 and #26, \( \lambda = 1 \) is a double root for the characteristic polynomial of the matrix \( A \) given by \( A = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix} \).

25. Which one of the following vectors is an eigenvector for the eigenvalue \( \lambda = 1 \)?

(a) \((3, -1)^T\)  
(b) \((3, 1)^T\)  
(c) \((1, -3)^T\)  
(d) \((-1, -3)^T\)

(e) none of the above

26. Which one of the following vector valued functions is the general solution of the linear equation \( y' = Ay \)?

(a) \( y = e^{t}(at + b)(1, 3)^T + be^{t}(0, 1)^T \)  
(b) \( y = e^{t}(a + b)(1, 3)^T + bte^{t}(0, 1)^T \)

(c) \( y = e^{t}(a + bt)(1, 3)^T \)  
(d) \( y = e^{t}(a + bt)(1, 3)^T + be^{t}(0, 1)^T \)

(e) none of the above
27. Consider the second order linear ODE \( y'' - 3y' - 10y = 21e^{-2t} - 50 \)

(a). Find a particular solution \( y_p \) for the given equation.
(b). Find the general solution \( y_h \) of the associated homogeneous equation.
(c). Find the general solution \( y \) for the given equation.
(d). Solve the initial value problem for the given equation with the initial conditions \( y(0) = 0 \) and \( y'(0) = 0 \).
28. A 200 gal tank initially contains 100 gal of pure water. Salt water solution containing 1 lb of salt for each gallon of water begins entering the tank at a rate of 2 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing salt water solution to leave the tank at a rate of 2 gal/min. Establish an initial value problem and then solve it for the salt content (lb) in the tank.
29. Consider the planar system \[
\begin{align*}
x' &= 2x + x^2 - xy \\
y' &= 4y + xy + y^2
\end{align*}
\]
(a) Find all \(x\)-nullclines for the system.
(b) Find all \(y\)-nullclines for the system.
(c) Find all equilibrium points for the system.
(d) Find the linearization of the system for each equilibrium point.
(e) Classify each equilibrium point (spiral sink, nodal source, etc.)
30. Consider the matrix \( A = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix} \)

(a) Find the characteristic polynomial of the matrix \( A \).
(b) Find all of the eigenvalues of \( A \).
(c) For each eigenvalue \( \lambda \), find an associated nonzero eigenvector.
(d) Find a fundamental set of real valued solutions for the system \( y' = Ay \).
(e) Calculate the Wronskian of the fundamental set of solutions from (d).
(f) Find the general solution of the system \( y' = Ay \).
(g) Solve the initial value problem \( y' = Ay \) with \( y(0) = (0, 2)^T \).