Final Exam MA 424/624

Please Show All Work

1. A population, $P$, is growing at a rate proportional to the current population. Let $r = 0.1$ be the growth rate (years$^{-1}$) and $t$ be in years.
   (a) Write a D.E for the population $P$ (2 points)
   (b) Solve the D.E. for $P$ given an initial population $P(0) = 320$ (4 points)
   (c) Find the number of years it will take for the population to triple (3 points)

2. Find the solution to the following D.E. using the method of integrating factors
   
   \[ t \frac{dy}{dt} + ty = t - y, \quad t > 0, \quad y(1) = 1 \]

   (a) Rewriting Equation (2 points)
   (b) Determine integrating factor, $\mu(t)$ (2 points)
   (c) Solving the D.E. using method of integrating factors (5 points)

3. For the given D.E. below, answer the following:
   \[ y' + \frac{2}{3-t}y = \frac{1}{t}, \quad y(1) = 1 \]

   (a) What is the order of the D.E? (1 point)
   (b) Is the D.E. linear? (1 point)
   (c) On what interval for $t$ can we ensure there exists a unique solution to the above D.E.
      (Make sure to identify and describe the continuity of $p(t)$ and $g(t)$) (5 points)

4. For the given D.E. below, answer the following:
   \[ yy'' + (y')^2 = 0, \quad t > 0, \quad y(1) = 1 \]

   (a) Is the D.E. linear? (1 point)
   (b) What is the order of the D.E.? (1 point)
   (c) Verify that $y_2(t) = t^{1/2}$ is a solution (2 points)
   (d) Given that $y_1(t) = 1$ is a solution to the above D.E., find $W(y_1, y_2)(t)$ (2 points)
   (e) Explain if/when/where/why we can write $y = c_1y_1 + c_2y_2$ based on your answer in (d)
      and the D.E. and ICs (3 points)

5. Find the solution to the following D.E. using the method of undetermined coefficients
   \[ y'' + 2y' - 6y = e^t, \quad y(0) = 0, \quad y'(0) = \frac{2}{3} \]

   (a) Determine $y_h$ (4 points)
   (b) Determine $Y_P$ (3 points)
   (c) Solution $y$ using ICs to determine constants (4 points)
6. For the dfield plot in Figure 1, answer the following:
   (a) What are the fixed points? (3 points)
   (b) Draw the corresponding phase line (dy/dt vs y) (3 points)
   (c) Classify each of the fixed points as stable, unstable, or semistable (3 points)
   (d) Write down a D.E. for the given dfield plot (3 points) (Hint: think about the graph of a general cubic function and how the coefficient of the highest power determines long term behavior)
   (e) This model is Logistic growth with a threshold. In terms of extinction and reaching a carrying capacity, explain what happens to the population if you start out in Region 1, Region 2, and Region 3. (3 points)

![Figure 1: dfield plot](image)

7. Find a series solution of the form $y = \sum_{n=0}^{\infty} a_n (x - x_o)^n$ using $x_o = 0$ for the following D.E.
   \[ y'' + xy' + 2y = 0, \quad y(0) = 3, \quad y'(0) = 2 \] (5)
   (a) Rewriting the D.E. using the series form of $y$ (3 points)
   (b) Equate coefficients of like powers of $x$ for powers 0 to 5 and determine how to rewrite coefficients $a_i$ in terms of $a_o$ and $a_1$ (5 points)
   (c) Write the solution as $y = a_o y_1(x) + a_1 y_2(x)$ (3 points)
   (d) Use ICs to determine solution $y$ (2 points)
8. Using the Laplace transform, find the solution to the following D.E.

\[ y'' + y = t, \quad y(0) = 1, \quad y'(0) = 0 \]  

(a) Finding Laplace transform of D.E. (3 points)  
(b) Rewriting and Partial Fractions (4 points)  
(c) Inverse Laplace Transform (3 points)  

9. Using the system of equations given below, answer the following:

\[ x' = x - 2y \]  
\[ y' = 3x - 4y \]  

(a) Rewrite in the form \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) (3 points)  
(b) Determine the eigenvalues (4 points)  
(c) Determine the corresponding eigenvectors (4 points)  
(d) Determine the general solution (3 points)  
(e) Determine the stability of the origin and type of node. (Short explanation based on eigenvalues or on the determinant and trace of \( \mathbf{A} \)) (2 points)  

10. Using the system of equations given below, answer the following:

\[ x' = -x + x^3 \]  
\[ y' = -2y \]  

(a) Find the critical points (Hint: there are 3) (4 points)  
(b) Use linearization theory to classify each of the critical points as a saddle, stable node, or unstable node. (i.e. look at Jacobian matrix evaluated at the critical points, you may discuss classification based on eigenvalues or determinant and trace of the matrix) (5 points)  
(c) Sketch a phase plane diagram (4 points)  

11. Determine the lower bound of the radius of convergence for a series solution about a point \( x_o = 2 \) for the following D.E. (4 points)

\[(1 + x^2)y'' + 4xy' + y = 0\]  

12. A spring-mass system is governed by the following D.E.

\[ u'' + 4u = 0.5\cos(t) \]  

(a) Is the system damped or undamped? (2 points)  
(b) If you were going to use the method of undetermined coefficients, what would your assumed form of \( Y_p \) be? (2 points)  
(c) Find \( y_h \) (5 points)