Rogue Waves: Refraction of Gaussian Seas and Rare Event Statistics

Eric J. Heller (Harvard University)
Lev Kaplan (Tulane University)

Talk outline:

- Introduction: what are rogue waves
- Synthesis of ***refractive*** and ***stochastic*** models
- How caustics form: refraction from current eddies
- Analogies with electron flow and other physical systems
- Smearing of caustics for stochastic incoming sea
- Quantifying residual effect of caustics: the “***freak index***”
- Wave height statistics: numerical and analytical results
- Questions and future directions

Introduction: Rogue Waves

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- About 10 large ships lost per year to presumed rogue waves – usually no communication
- Also major risk for oil platforms in North Sea, etc.
- Probability seems to be much higher than expected from Gaussian random model of wave heights
- Large rogue waves have height of 30 m or more, last for minutes or hours
- Long disbelieved by oceanographers, first hard evidence in 1995 (North Sea)
- Tend to form in regions of strong current: Agulhas, Kuroshio, Gulf Stream

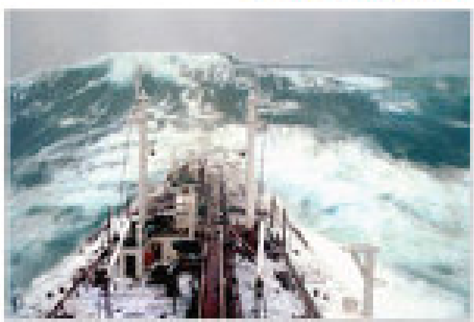
Introduction: Rogue Waves

Rogue Giants at Sea

By WILLIAM J. BROAD
Published: July 11, 2006

The storm was nothing special. Its waves rocked the Norwegian Dawn just enough so that bartenders on the cruise ship turned to the usual palliative — free drinks.

Enlarge this Image



Karsten Petersen

STORM SURGE The chief engineer of the Stolt Surf took photographs as the tanker met a rogue wave in 1977. The deck, nearly 75 feet above sea level, was submerged.

Then, off the coast of Georgia, early on Saturday, April 16, 2005, a giant, seven-story wave appeared out of nowhere. It crashed into the bow, sent deck chairs flying, smashed windows, raced as high as the 10th deck, flooded 62 cabins, injured 4 passengers and sowed widespread fear and panic.

"The ship was like a cork in a bathtub," recalled Celestine Mcelhatton, a passenger who, along with 2,000 others, eventually made it back to Pier 88 on the Hudson River in Manhattan. Some vowed never to sail again.

Synthesis of refractive and stochastic models

- Three common approaches to rogue wave formation:
  - Stochastic: unlucky constructive addition of Longuet-Higgins Gaussian random waves
  - Refractive: focusing by current eddies (Peregrine, White & Fornberg, …)
  - Nonlinear growth (Trulse & Dysthe, Onorato et al, …)
- Difficulties:
  - Stochastic: extreme events too rare
  - Refractive: ignores randomness in incoming sea
  - Nonlinear: desirable to have trigger

Synthesis of refractive and stochastic models

- Our approach:
  - Combine stochastic and refractive models by analyzing effect of caustics on random incoming sea
  - Allows many more extreme events than in pure stochastic model
  - Deals with issue of sensitivity to initial conditions
    - Allows for *statistical* description of rogue waves
  - Nonlinearity currently absent
    - Events we predict can be regarded as input to full nonlinear theory

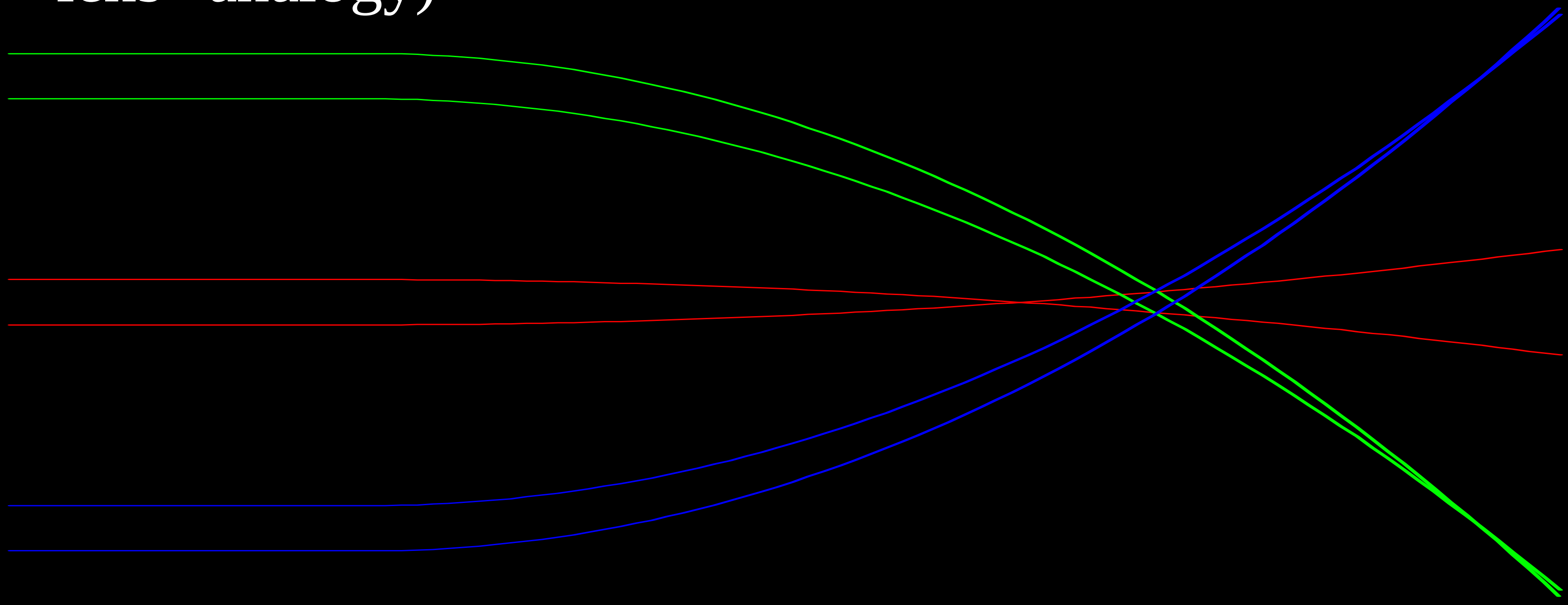
Ray picture ("semiclassical")

- Consider rays moving through weakly scattering non-uniform medium, in *y*-direction
  - Phase space coordinates: transverse position x and transverse wave vector k_x
  - Initial condition: unidirectional rays (k_x=0, uniform x)
  - Evolution: $$\frac{dk_x}{dt} = -\frac{\partial\omega}{\partial x} \qquad \frac{dx}{dt} = \frac{\partial\omega}{\partial k_x}$$
  - For deep water surface gravity waves with current

$$\omega(\vec{r},\vec{k}) = \sqrt{gk} + \vec{u}(\vec{r})\cdot\vec{k}$$

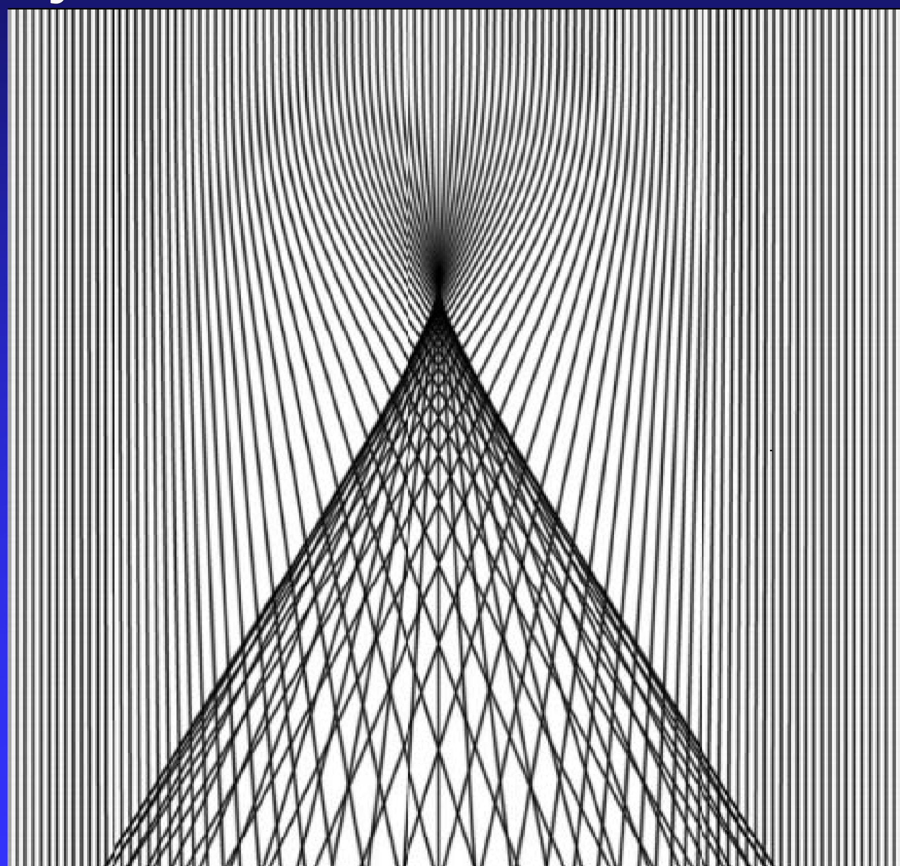
How caustics form: refraction from current eddies

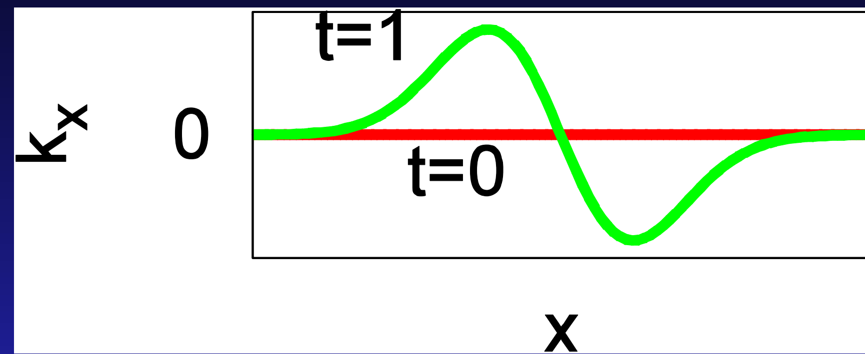
- Parallel incoming rays encountering single eddy
- Eddy acts like potential dip in particle mechanics
- Focusing when all paths in a given neighborhood coalesce at a single point (caustic), producing infinite ray density
- Different groups of paths coalesce at different points ("bad lens" analogy)

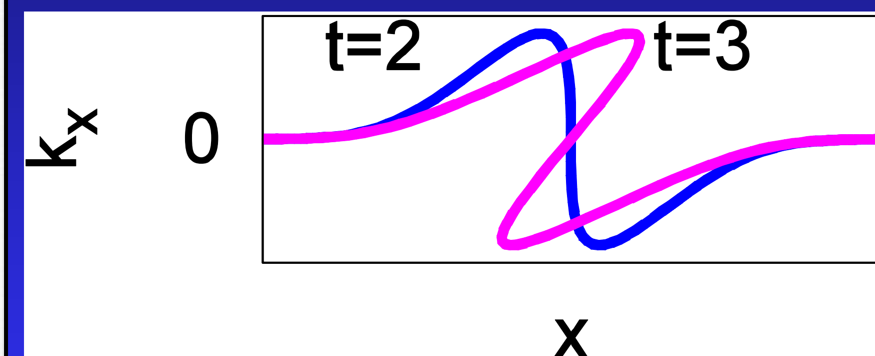


How caustics form: refraction from current eddies

Cusp singularity followed by two lines of fold caustics





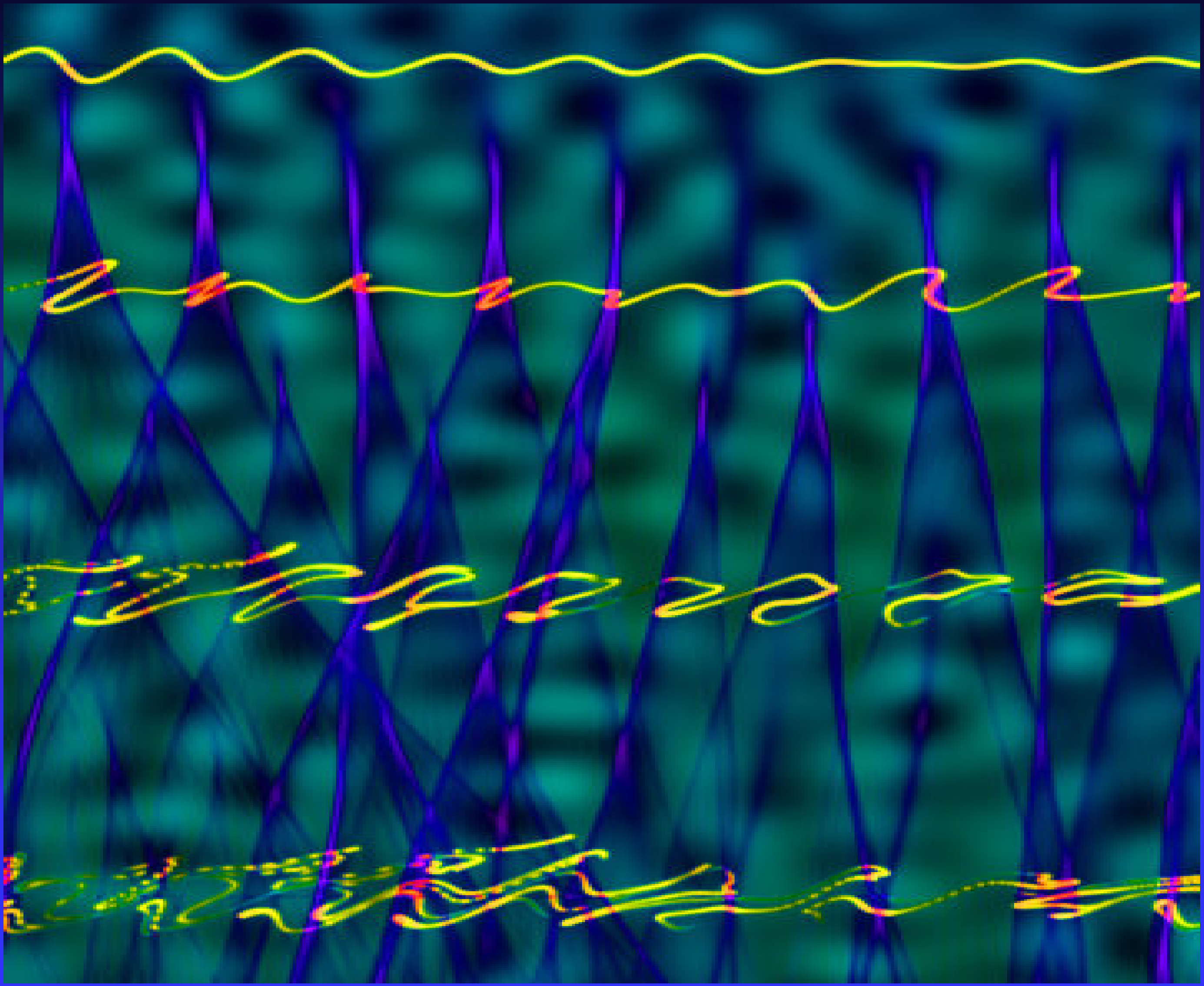


At each *y*, infinitesimally close paths near some *x* must coalesce

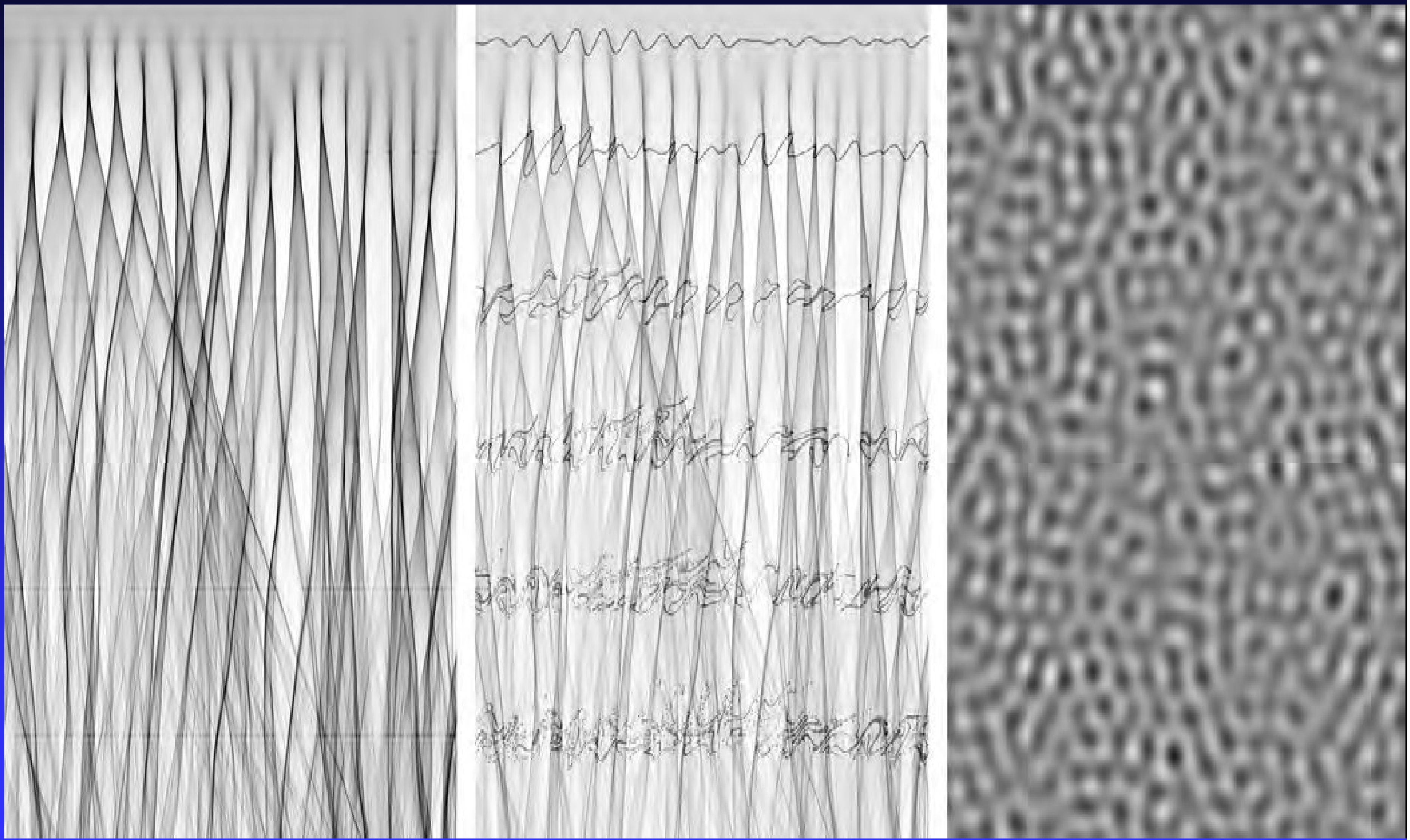
Refraction from weak, random currents

- Incoming wave with velocity v, wavelength λ
- Given random current field with velocity fluctuations $\delta u << v$ on distance scale $\xi >> \lambda$: small angle scattering
  - First singularities form after distance $d \propto (\delta u / v)^{-2/3} \xi$
- Further evolution: exponential proliferation of caustics
  - Tendrils decorate original branches
  - Universal branch statistics with single distance scale d
- Qualitative structure independent of
  - dispersion relation (e.g. $\omega \sim k^2$ for Schrödinger)
  - details of random current field

Multiple branching

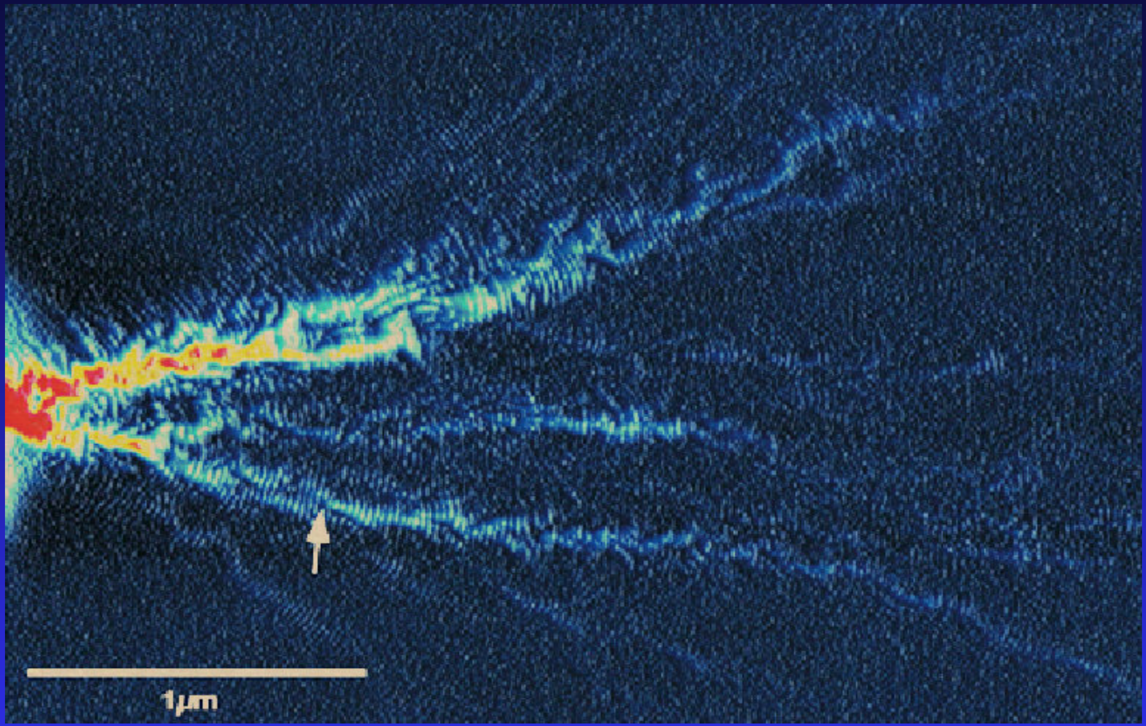


Multiple branching



Analogies with other physical systems

- Electron flow in nanostructures (10^{-6} m)



Topinka et al, Nature (2001)

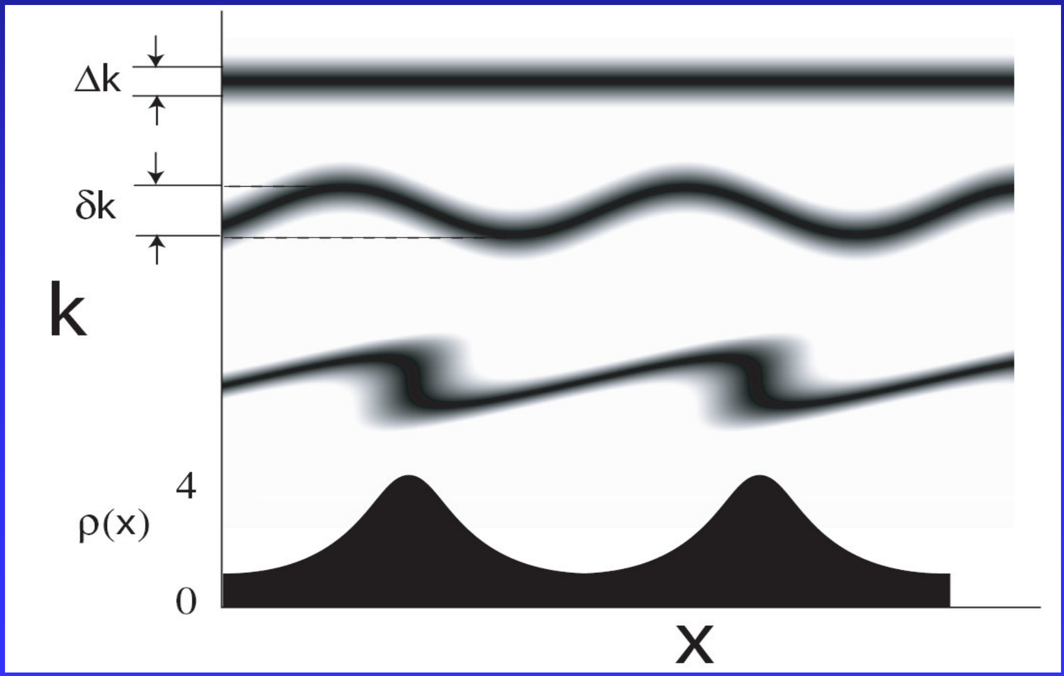
- Microwave resonators, Stöckmann (1 m)
- Long-range ocean acoustics, Tomsovic et al (20 km)
- Starlight twinkle, Berry (2000 km)
- Gravitational lensing, Tyson (10^9 light years)

Problems with refraction picture of rogue waves

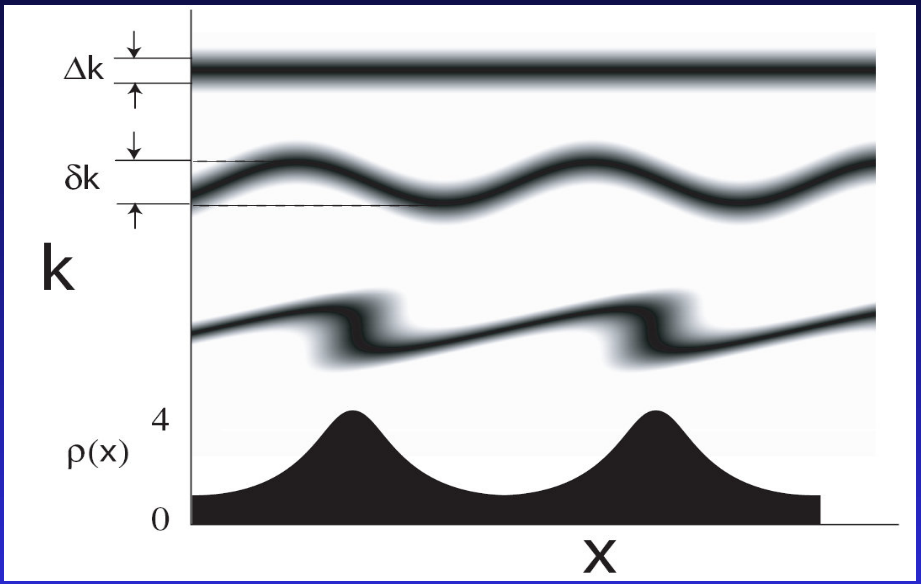
- Assumes single-wavelength and unidirectional initial conditions, which are unrealistic and unstable (Dysthe)
- Singularities washed out only on wavelength scale
- Predicts regular sequence of extreme waves *every time* incoming swell encounters variable current field
- No predictions for actual wave heights or probabilities
- Solution: replace incoming plane wave with random initial spectrum
  - Finite range of wavelengths and directions

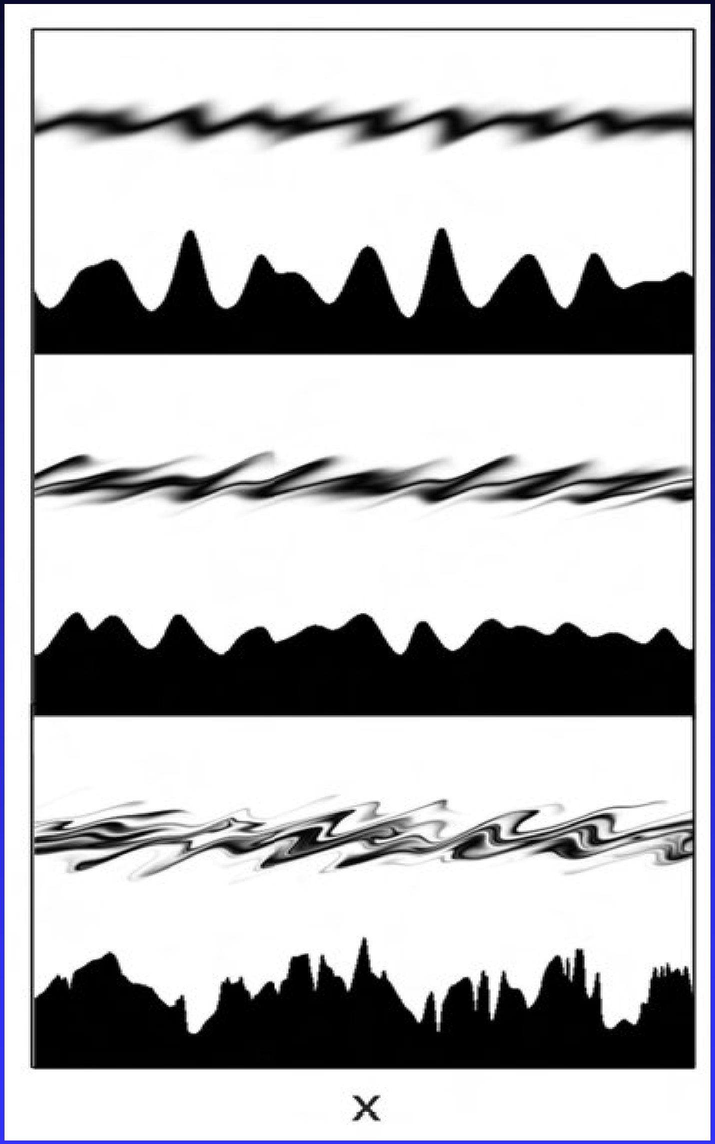
Smearing of caustics for stochastic incoming sea

- Singularities washed out by randomness in initial conditions
- “Hot spots” of enhanced average energy density remain as reminders of where caustics would have been



Smearing of caustics for stochastic incoming sea





Competing effects of focusing and initial stochasticity:

Δk_x = initial wave vector spread

δk_x = wave vector change due to refraction

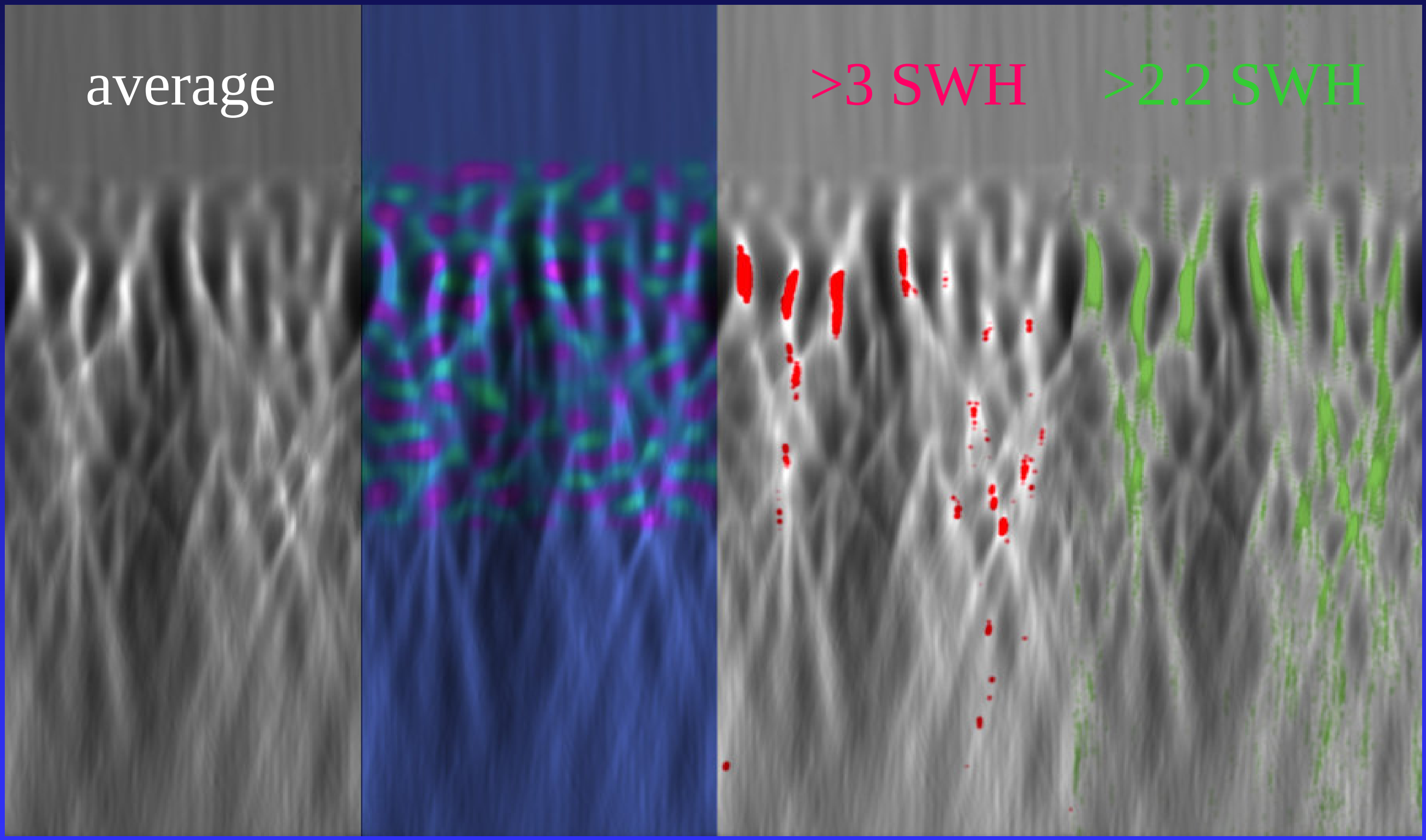
Quantifying residual effect of caustics: the "*freak index*"

- Define freak index $\gamma = \delta k_x / \Delta k_x$
- Equivalently $\gamma = \delta\theta / \Delta\theta$
  - $\Delta\theta = \Delta k_x / k$ = initial directional spread
  - $\delta\theta$ = typical deflection before formation of first cusp $\sim (\delta u / v)^{2/3} \sim \xi / d$
- Most dangerous: well-collimated sea impinging on strong random current field ($\gamma \geq 1$)
- Hot spots corresponding to ***first*** smooth cusps have highest energy density

$$\Delta\theta(y) \approx \Delta\theta\sqrt{1+\gamma^2(y/d)} \quad \Rightarrow \quad \gamma(y) \approx \sqrt{d/y}$$

Implications for wave height statistics

- Simulations using linear **Schrödinger** equation (long-time average & extreme events)



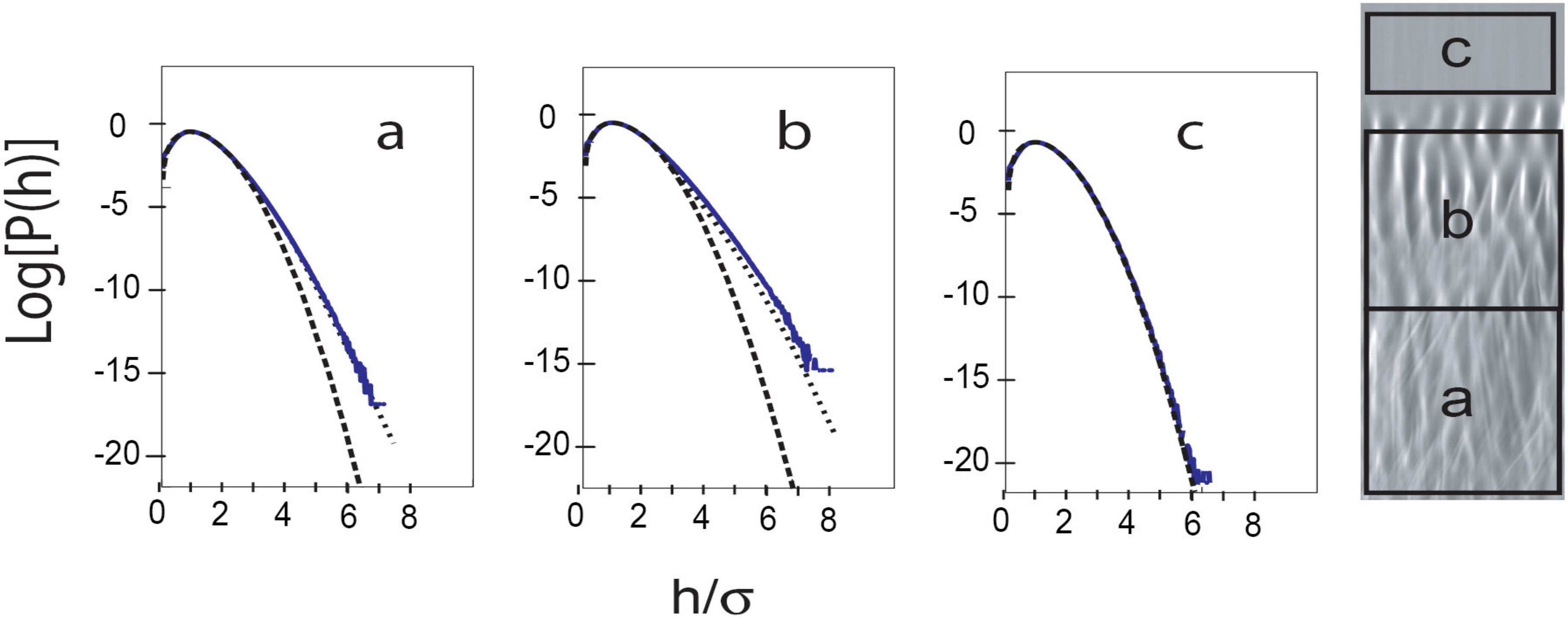
$\gamma \approx 2$

- 1 SWH=significant wave height $\approx 4\sigma$ crest to trough

Implications for wave height statistics

- Modified distribution of wave heights

$\gamma = 3.4$



Dashed = Rayleigh (Gaussian random waves)

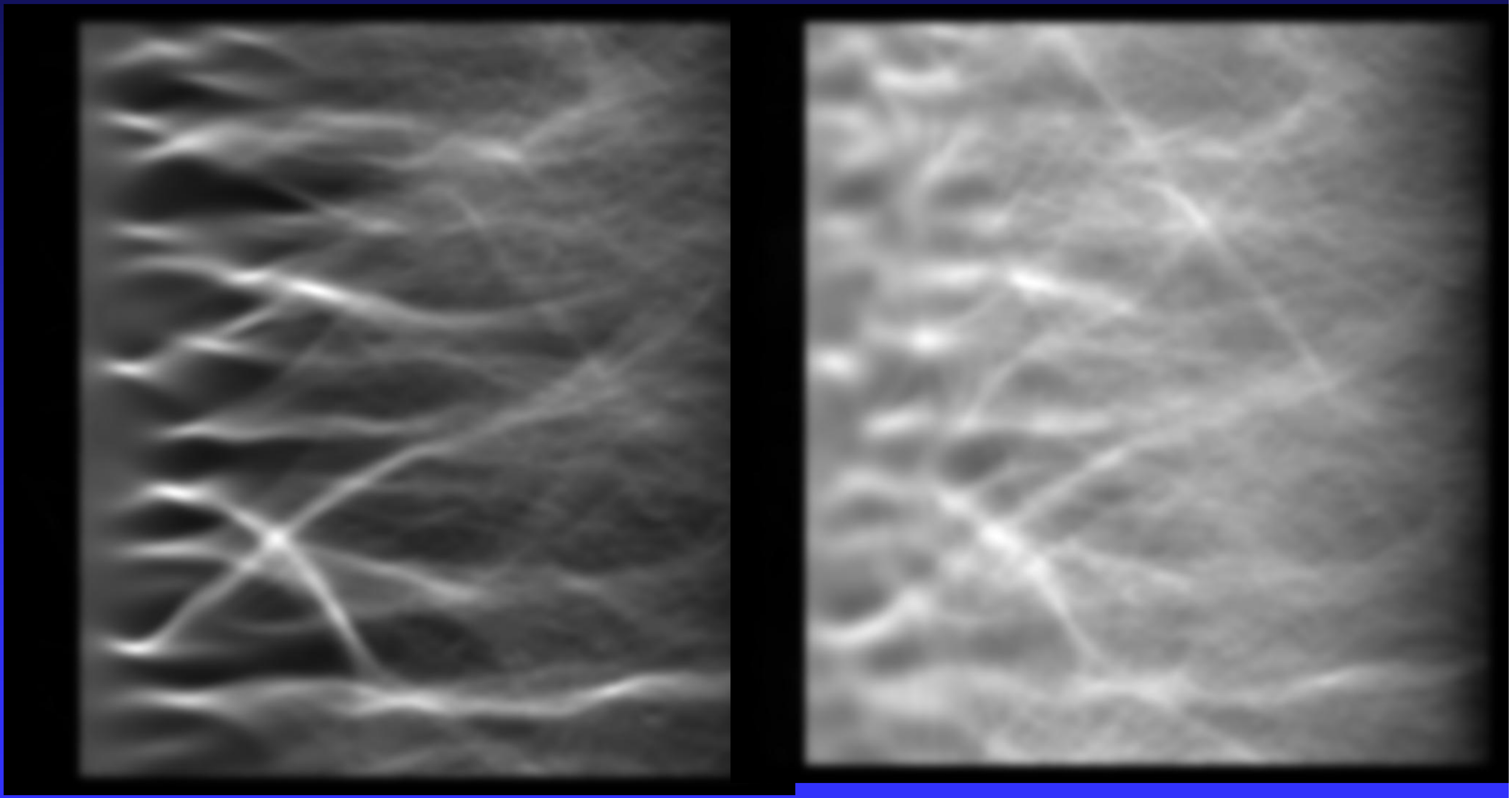
Dotted = Prediction based on *locally* Gaussian fluctuations around local intensity given by ray density

Simulations for ocean waves

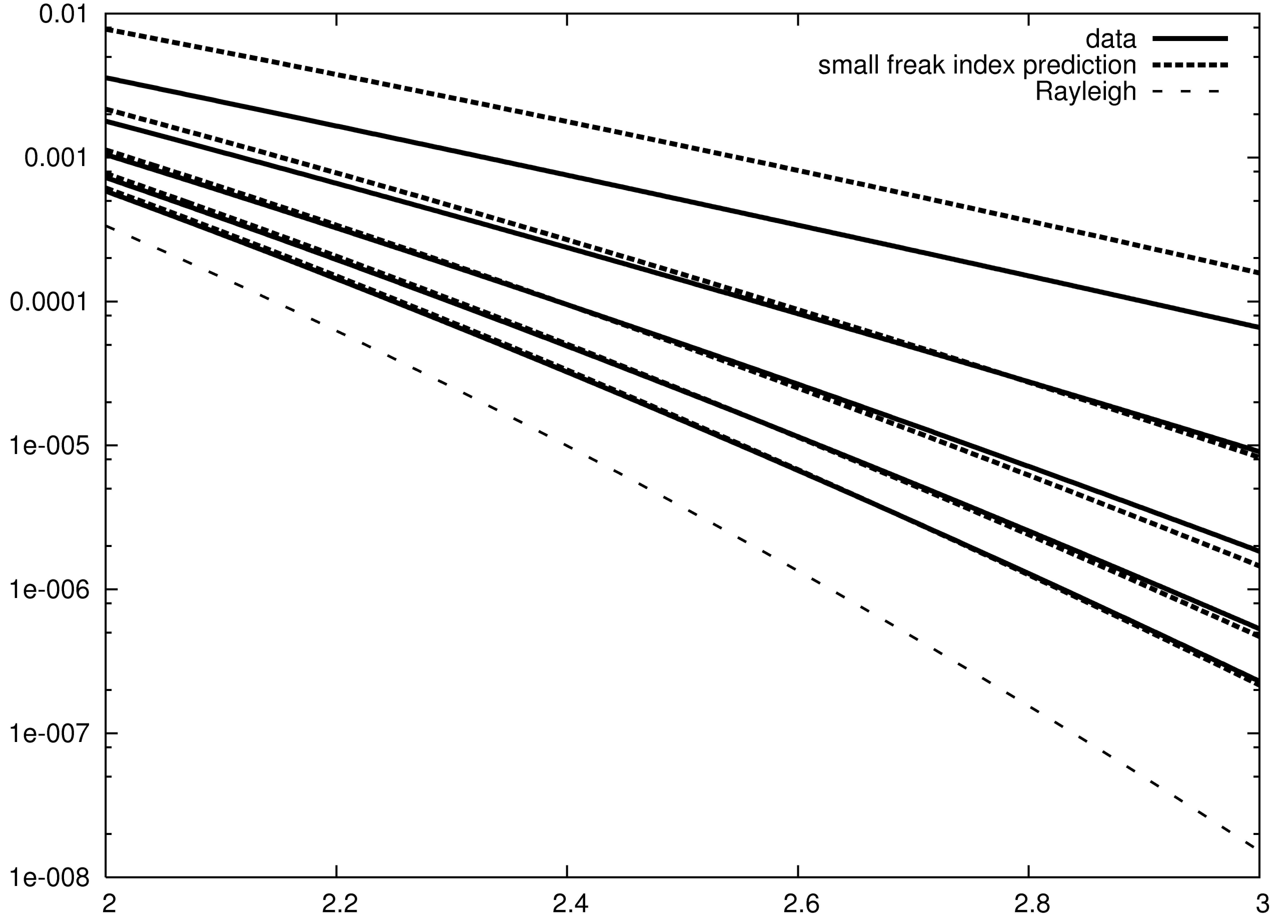
- Incoming sea with v=7.8 m/s (T=10 s, λ=156 m)
- Random current $\vec{u}\,(x,y) = \vec{\nabla} \times f\,(x,y)$
  with rms velocity u_{rms} = 0.5 m/s
- $f(x,y)$ Gaussian random with correlation ξ=20 km
- Dimensionless parameters:
  - $\lambda \,/\, \xi << 1$ ("semiclassical" limit)
  - $\delta\theta \sim (u_{\mathrm{rms}} \,/\, v)^{2/3} << 1$ (small-angle scattering)
  - $\Delta\theta$ = spreading angle = 5, 10, 15, 20, 25°
  - $\gamma = \delta\theta \,/\, \Delta\theta$ = freak index
- Calculate ray density, then assume locally Rayleigh behavior to obtain $P(h > x \bullet SWH)$

Typical ray calculation for ocean waves

$\Delta\theta = 5^{\circ}$ $\Delta\theta = 25^{\circ}$



Wave height distribution for ocean waves



Analytics: limit of small freak index

- Rayleigh: $P(h > x \bullet SWH) = \exp(-2x^2/\sigma^2)$
- Average: $P(h > x \bullet SWH) = \int \exp(-2x^2/\sigma^2) g(\sigma) d\sigma$
- γ<<1: $g(\sigma)$ Gaussian with mean 1 and small width $\delta \propto \gamma \propto 1/\Delta\theta$
- Stationary phase:

$$P(h > x \bullet SWH) = \sqrt{\frac{1+\varepsilon}{1+3\varepsilon}} \exp[-\varepsilon(1+\frac{3}{2}\varepsilon)/\delta^2]$$

where $\varepsilon(1+\varepsilon)^2 = 2\delta^2 x^2$

- Perturbative expansion: for $\varepsilon << 1$

$$P(h > x \bullet SWH) = [1 + 2\delta^2(x^4 - x^2)] \exp(-2x^2)$$

Summary

- Refraction of stochastic Gaussian sea produces lumpy energy density
  - Skews formerly Rayleigh distribution of wave heights
- Significant energy lumps may survive averaging over initial wave direction & wavelength
  - despite chaoticity displayed by individual ray trajectories
- Overall wave height distribution given by averaging:
  - SWH, low-order moments effectively unchanged
  - Probability of extreme waves enhanced dramatically
- Importance of refraction quantified by freak index γ
  - Spectacular effects in tail even for small γ
- Refraction may serve as trigger for full non-linear evolution

Questions and future directions

- JONSWAP incoming spectrum
- ***Nonlinear*** evolution of lumpy energy landscape
  - Second-order wave theory (Rayleigh $\rightarrow$ Tayfun)
  - Higher-order effects (full wave equation)
  - Nonlinear spreading/defocusing or nonlinear enhancement of rogue waves due to modulational instability? (Dysthe, Onorato, Osborne, Zakharov…)
- Movement of energy lumps due to changing eddy configuration
- Experimental detection of energy lumps?