A SPHERICAL BIPHASIC MODEL FOR RADIAL DEFORMATION IN A CHONDRON

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INTRODUCTION

The pericellular matrix (PCM) is a narrow region of tissue surrounding chondrocytes in articular cartilage and is believed to influence the mechanical environment of the cell. Compared to the extracellular matrix (ECM), the PCM contains type VI collagen and has a higher proteoglycan concentration. As material parameters for the chondron (cell+PCM) are obtained, biomechanical models can be developed to study the effect of the PCM on signal transmission to the cell. In a multiscale biphasic FEM model of deformation due to a step load, Guilak and Mow [2] demonstrated that the inclusion of a PCM layer at the microscale significantly altered the local mechanical environment of a single cell. Recently, a layered contact solution was used in a micropipette aspiration experiment to determine PCM elastic properties for intact chondrons extracted mechanically from human cartilage. The mean PCM Young's modulus was reported as 66.5KPa in the healthy group and 41.2KPa in the osteoarthritic group [1].

This study considers a simplified model for dynamic loading of a single chondron idealized as a spherical cell with attached PCM layer (Fig.1). Both regions are assumed to be linear isotropic biphasic continua in purely radial deformation. To date, only chondron elastic parameters have been measured. Hence, this study focuses on analyzing effects of the reportedly large PCM-to-cell modulus ratio on transmission of displacement through the PCM layer. Analytical and numerical solutions are employed, respectively, for the cases of uniform and non-uniform permeability in the chondron.

MODEL

The reduced biphasic governing equations are:

$$\partial_t u = \begin{cases} k_C H_C^A (\rho^{-2}\partial_\rho(\rho^2 \partial_\rho u) - 2\rho^{-2} u) & 0 \le \rho \le a \\ k_P H_P^A (\rho^{-2}\partial_\rho(\rho^2 \partial_\rho u) - 2\rho^{-2} u) & a < \rho \le b \end{cases}, \quad t > 0 \qquad (1)$$

$$p = \begin{cases} H_C^A (2\rho^{-1} u + \partial_\rho u) + f_C(t) & 0 \le \rho \le a \\ H_P^A (2\rho^{-1} u + \partial_\rho u) + f_P(t) & a < \rho \le b \end{cases}, \quad t > 0 \qquad (2)$$

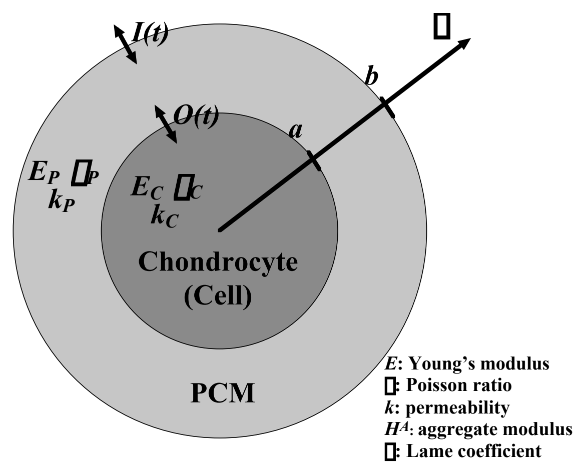

Figure 1. Spherical model of a chondron

We model deformation due to a dynamic sinusoidal displacement input (amplitude u_0, forcing freq. ω). The boundary conditions are:

$$u(0,t) = 0, \qquad u(b,t) = I(t) = u_0 \sin \omega t \qquad (3)$$

At the cell-PCM interface, the interface conditions of continuous displacement and traction reduce to:

$$[[u(a,t)]] = 0, \qquad [[H^A \partial_\rho u(a,t)]] = 2a^{-1}(\lambda_C - \lambda_P) u(a,t) \qquad (4)$$

The two arbitrary functions in (2) can be used to satisfy pressure continuity at $\rho = a$ and to match to an applied pressure at $\rho = b$. The pressure and displacement fields uncouple and equations (1),(3-4) yield a signal transmission model that was used to determine the scaled amplitude of the transmitted signal $O(t) = u(a,t)/u_0$.

Uniform permeability: analytical series solution

In the case $k_C = k_P$, (3-4) give rise to an eigenvalue problem that admits real eigenvalues and orthogonal eigenfunctions[3]. Condition

(3b) is incorporated into the solution via Duhamel's principle. The following series solution for chondron displacement results:

$$\frac{u(\xi,t)}{u_0} = \sum_{j=1}^{\infty} \left[K_j(\omega) e^{-\lambda_j t} + L_j(\omega)\sin\omega t - K_j(\omega)\cos\omega t \right] \phi_j(\xi) + M(\xi,t) \quad (5)$$

$$\text{where: } M(\xi,t) = \sin\omega t \begin{cases} \alpha_3 \xi^2 & 0 \le \xi \le a \\ \alpha_1 \xi^2 + \alpha_2 \xi & a < \xi \le b \end{cases}$$

The eigenfunctions in (5) are:

$$\phi_j = \begin{cases} (r_j\xi)^{-2}[\sin(r_j\xi) - r_j\xi\cos(r_j\xi)] & 0 \le \xi \le a \\ \gamma_j (s_j\xi)^{-2}[\sin(s_j(\xi+\delta_j)) - s_j\xi\cos(s_j(\xi+\delta_j))] & a < \xi \le b \end{cases} \quad (6)$$

where $r_j = \lambda_j^{1/2}(kH_C^A)^{-1/2}$ and $s_j = \lambda_j^{1/2}(kH_P^A)^{-1/2}$. The constants $\lambda_j, \gamma_j, \delta_j$ are eigenvalues that can be determined from a single reduced characteristic equation in λ_j. A detailed derivation of (5-6) with formulae for $K_j(\omega), L_j(\omega), \alpha_1, \alpha_2, \alpha_3$ can be found in [3].

Non-uniform permeability: finite difference solution

In the case $k_C \ne k_P$, equations (1) ,(3-4) do not admit an analytical solution and were solved numerically. A finite difference method was used to discretize (1) with a forward difference in time and a centered difference in space. A regular spatial mesh was chosen such that no point was coincident with $\xi = a$. Condition (4a) was enforced by equating displacements at the two mesh points on either side of the interface. Condition (4b) was enforced on either side of the interface using a first-order difference, thus ensuring that no derivatives were taken across the interface. For the results presented, 300 spatial points and 3000 time points were sufficient to compute steady-state displacement amplitudes to two decimal places. In the case of uniform permeability, the numerical solution was in agreement with (5).

RESULTS

Characteristic diffusion time for the chondron

In the case $k_C = k_P$, an asymptotic analysis of the characteristic equation for a stiff PCM yielded the relation $t_P = (0.04953)t_C$ [3], where $t_C = a^2/(kH_C^A)$ is the biphasic gel relaxation time for the chondrocyte. Our model indicates that the presence of a stiff PCM layer reduces the biphasic gel relaxation time of the chondron by a factor of 20. Hence, a stiff PCM dissipates transient deformation components on time scales that are rapid compared to the biphasic gel relaxation time of the cell alone.

Parametric analysis of steady-state displacement amplitude

As a measure of mechanical signal transmission in the chondron, we conducted a parametric analysis of the scaled steady-state amplitude at the cell-PCM interface $O(t) = u(a,t)/u_0$ that results after an interval of 10secs(Fig. 2). The geometry was set at $a = 10\mu$m, b $= 12.5\mu$m and fixed material properties were $\nu_C = 0.45$, $\nu_P = 0.1$, $E_C = 1$KPa and $k_C = 10^{-15}\text{m}^4(\text{Ns})^{-1}$.The applied displacement amplitude in (3b) was taken as $u_0 = 0.1(b - a)$. Amplitude was determined for 20 forcing frequencies(0-3Hz) with variation of the parameters E_P and k_P. For uniform permeability on the order typically associated with human articular cartilage, we observed that an increasingly large ratio of PCM-to-cell Young's modulus enhanced signal transmission to a near optimal state (Fig. 2a). We also observed a significant decrease in transmission amplitude for the reported mean osteoarthritic modulus [1], particularly at higher frequencies. Increasing the permeability of the PCM to 100 times that of the cell enhanced signal transmission and reduced differences between the healthy and OA cases(Fig. 2b). Transmission was greatly diminished when the PCM permeability was 100 times smaller than that of the cell (Fig. 2c). As experimental measurements of chondron permeability become available, we will develop more elaborate signal transmission models.

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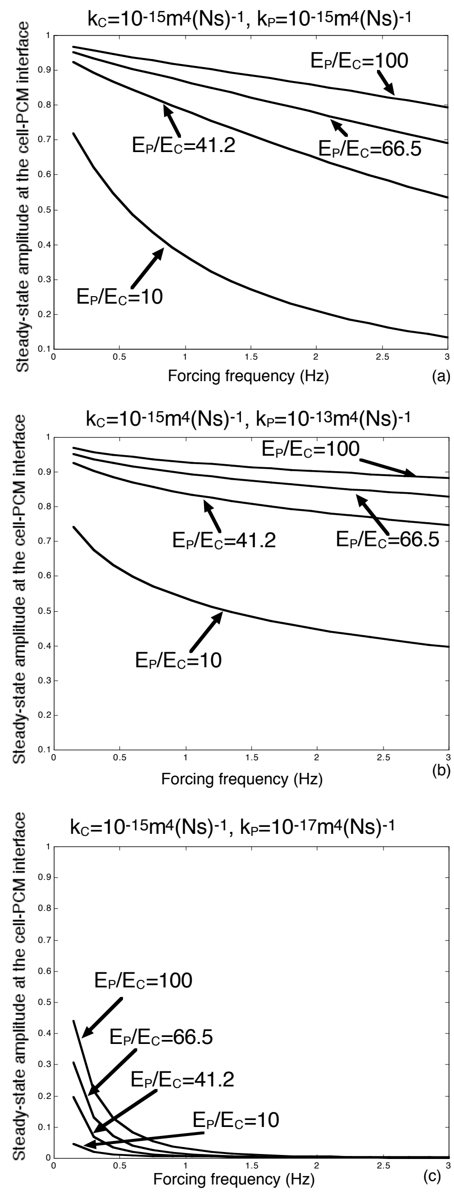

Figure 2. Parametric analysis of displacement signal transmission at steady-state