A MULTIDOMAIN FINITE ELEMENT METHOD TO SOLVE THE NON LINEAR ONE-DIMENSIONAL EQUATIONS OF BLOOD FLOW IN DEFORMABLE VESSELS

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INTRODUCTION

Blood flow in the arterial system is modeled by solving the governing equations on a domain and prescribing boundary conditions to account for the vessels downstream. These outflow boundary conditions have been generally limited to prescribed velocity, pressure or traction. The limitation of this approach lies in the fact that this necessitates *a priori* knowledge of the distribution of the blood flow or the outlet pressure for the branch vessels. This information is rarely known in most practical situations and depends upon the solution within the domain of interest. Olufsen has described a finite difference method for coupling a numerical method with impedance derived from a fractal tree [1]. Others have adopted an iterative coupling approach.

We have developed a space-time finite element method for solving the one-dimensional equations of blood flow including a resistance and an impedance boundary condition based on fractal trees [2,3]. However the impedance boundary condition assumes flow periodicity and thus cannot model dynamic changes downstream such as occur in the coronary bed. We describe a new approach for specifying outflow boundary conditions using a “coupled multidomain method" that provide a mathematical framework to develop several types of boundary conditions [4].

METHOD

Governing equations and finite element formulation

The flow is assumed to be Newtonian, incompressible and with a time-varying parabolic flow profile. The primary variables are the volumetric flow rate Q and the cross-sectional area S, which is related to the pressure through a constitutive relationship for an elastic domain:

$$p(z,t) = \tilde{p}\left(S(z,t), z, t\right)$$

The constant density and kinematic viscosity of the fluid are given by ρ and ν, the external force by f, and ψ is an outflow function. The continuity and momentum balance equations are integrated and written as a system of quasi-linear partial differential equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial z} - \boldsymbol{K}\frac{\partial^2 \boldsymbol{U}}{\partial z^2} = \boldsymbol{G} \qquad \text{where}$$

$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} S \\ Q \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} U_2 \\ (1+\delta)\dfrac{U_2{}^2}{U_1} + \dfrac{1}{\rho}\Phi \end{bmatrix} \qquad \Phi = \int_{p_0}^{p(z,t)} \tilde{S}(\mathsf{p}, z, t)\, d\mathsf{p}$$

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 \\ 0 & \nu \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} -\psi \\ Sf + N\dfrac{U_2}{U_1} + \displaystyle\int_{p_0}^{p} \frac{1}{\rho}\frac{\partial \tilde{S}(\mathsf{p}, z, t)}{\partial z}\, d\mathsf{p} \end{bmatrix} \qquad \delta = \frac{1}{3} \qquad N = -8\pi\nu$$

We use initial conditions for S and Q, a prescribed inlet flow rate, and outlet boundary conditions discussed subsequently.

The weak formulation of the initial boundary value problem is given as follows: find $\boldsymbol{U}$ such that for every weighting function $\boldsymbol{W}=[W_1, W_2]^{\mathrm{T}}$,

$$\int_0^T \int_0^L \left(-\boldsymbol{W}_{,t}^{\boldsymbol{T}}\boldsymbol{U} - \boldsymbol{W}_{,z}^{\boldsymbol{T}}\boldsymbol{F} + \boldsymbol{W}_{,z}{}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{U}_{,z} - \boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{G}\right)\, dz \;\; dt + \int_0^T \left[\boldsymbol{W}^{\boldsymbol{T}}\left(\boldsymbol{F} - \boldsymbol{K}\boldsymbol{U}_{,z}\right)\right]_0^L dt$$

$$+\int_0^L \boldsymbol{W}^{\boldsymbol{T}}(z,T)\;\; \boldsymbol{U}(z,\mathrm{T})\;\; dz - \int_0^L \boldsymbol{W}^{\boldsymbol{T}}(z,0)\;\; \boldsymbol{U}^0(z)\;\; dz = 0$$

Multiscale Method

We adopt a multiscale approach [5] to derive appropriate outflow boundary conditions. We divide the spatial domain Ω=[0,L] into an upstream “numerical” domain $\hat{\Omega}$=[0,B], and a downstream “analytic” domain Ω'=[B,L], separated by the cross sectional area at B. We define a disjoint decomposition of our variables and weighting functions, for example for our unknown solution vector, $\boldsymbol{U}$,

$$\boldsymbol{U} = \hat{\boldsymbol{U}} + \boldsymbol{U}', \quad \hat{\boldsymbol{U}}\Big|_{\Omega'} = \boldsymbol{0} \;\; and \;\; \boldsymbol{U}'\Big|_{\hat{\Omega}} = \boldsymbol{0}$$

We define the operators $\boldsymbol{M}$ and $\boldsymbol{H}$ on the Ω' domain based on the model of the downstream domain

$$\left[\boldsymbol{F}(\boldsymbol{U}') - \boldsymbol{K}\boldsymbol{U}'_{,z}\right]_{z=B} = \left[\boldsymbol{M}(\boldsymbol{U}') + \boldsymbol{H}\right]_{z=B}$$

We insert these expressions into our variational form enforcing the continuity of the trial and test functions at the interface. We thus obtain the original variational form specialized to the 1D numerical domain $\hat{\Omega}$, with the addition of a boundary term (boxed below) accounting for the interface to the 1D analytic domain,

$$\int_0^T\int_0^B\left(\hat{\boldsymbol{W}}_{,t}^T\hat{\boldsymbol{U}}+\hat{\boldsymbol{W}}_{,z}^T\boldsymbol{F}(\hat{\boldsymbol{U}})-\hat{\boldsymbol{W}}_{,z}{}^T\boldsymbol{K}\hat{\boldsymbol{U}}_{,z}+\hat{\boldsymbol{W}}^T\boldsymbol{G}(\hat{\boldsymbol{U}})\right)\ dz\ \ dt$$
$$-\int_0^L\hat{\boldsymbol{W}}^T(z,T)\ \hat{\boldsymbol{U}}(z,\mathrm{T})\ dz+\int_0^L\hat{\boldsymbol{W}}^T(z,0)\ \hat{\boldsymbol{U}}(z,0)\ dz$$
$$+\int_0^T\left[\hat{\boldsymbol{W}}^T\left(\boldsymbol{F}(\hat{\boldsymbol{U}})-\boldsymbol{K}\hat{\boldsymbol{U}}_{,z}\right)\right]_{z=0}dt+\boxed{\int_0^T\left[\hat{\boldsymbol{W}}^T\left(\boldsymbol{M}(\hat{\boldsymbol{U}})+\boldsymbol{H}\right)\hat{n}\right]_{z=B}dt}=0$$

Space-Time Derivation

We follow the general Space-Time method [2] with piecewise constant in time shape functions. Thus, in the time slab from t_n to t_{n+1},

$$\Delta t_n\int_0^B\left(\hat{\boldsymbol{W}}_{,z}^{T,n+1}\boldsymbol{F}^{n+1}(\hat{\boldsymbol{U}}^{n+1})-\hat{\boldsymbol{W}}_{,z}{}^{T,n+1}\boldsymbol{K}\hat{\boldsymbol{U}}_{,z}^{n+1}+\hat{\boldsymbol{W}}^{T,n+1}\boldsymbol{G}^{n+1}(\hat{\boldsymbol{U}}^{n+1})\right)\ dz$$
$$-\int_0^B\hat{\boldsymbol{W}}^{T,n+1}\left(\hat{\boldsymbol{U}}^{n+1}-\hat{\boldsymbol{U}}^{n}\right)dz+\Delta t_n\left[\hat{\boldsymbol{W}}^{T,n+1}\left(\boldsymbol{F}^{n+1}(\hat{\boldsymbol{U}}^{n+1})-\boldsymbol{K}\hat{\boldsymbol{U}}_{,z}^{n+1}\right)\right]_{z=0}$$
$$+\boxed{\int_{t_n^+}^{t_{n+1}^-}\left[\hat{\boldsymbol{W}}^{T,n+1}\left(\boldsymbol{M}^{n+1}(\hat{\boldsymbol{U}})+\boldsymbol{H}^{n+1}\right)\hat{n}\right]_{z=B}dt}=0$$

We will look more precisely at the last term but first we define:

$$\mathrm{B}_i=\int_{t_n}^{t_{n+1}}\left[\hat{W}_i\left(M_i\left(\hat{U}\right)+\hat{H}_i\right)\hat{n}\right]_{z=B}dt\quad i=1,2$$

RESULTS: MODELING THE DOWNSTREAM DOMAIN

We can use various representations of the downstream domain, depending on the importance of the vasculature involved. We neglect nonlinearities and the longitudinal viscous force to obtain:

$$Q'(B,t)=\left[M_1(U_1',U_2')+H_1\right]_{z=B}$$
$$\Phi'=\int_{p_0'}^{p'(z,t)}\tilde{S}(p,z,t)dp=\rho\left[M_2(U_1',U_2')+H_2\right]_{z=B}$$

M_2 and H_2 depend essentially on the constitutive equation in the downstream domain. With a linear constitutive relationship,

$$p-p_0=\frac{1}{C_p}(S-S_0)\qquad;\quad M_2(S')=\frac{S'^2}{2\rho C_p}\qquad H_2=-\frac{S_0^{\ 2}}{2\rho C_p}$$

The operators M_2 and H_2 for every boundary condition are influenced only by functions at the present time and the initial conditions. Thus:

$$\mathrm{B}_2=\Delta t_n\hat{W}_2^{n+1}(B)\left[\frac{S^{n+1^2}-S_0^{\ 2}}{2\rho C_p}\right]_{z=B}\hat{n}$$

Instantaneous case

The Dirichlet (prescribed flow rate or pressure) and purely resistive boundary conditions depend only on the present time, e.g. in the resistive case,

$$M_1(S)=\frac{1}{R(t)}\tilde{p}\left(S(z,t),z,t\right)\qquad\qquad H_1=0$$

There is no memory for these boundary conditions. The flux term is discretized in time to obtain in time slab $n+1$:

$$\mathrm{B}_1=\Delta t_n*\left[\hat{W}_1^{n+1}M_1^{n+1}\left(\hat{S}^{n+1}\right)\right]_{z=B}\hat{n}$$

Memory case

More sophisticated boundary conditions include the Windkessel model, impedance based models where the downstream domain is approximated using a periodic one-dimensional linear wave propagation theory, and the more general damped wave equations (which do not assume periodicity). For brevity, we only provide here the example of the impedance boundary condition defined over one period T, with *y(z,t)* the inverse Fourier transform of the admittance,

$$M_1(S)=\frac{1}{T}\int_{t-T}^{t}\tilde{p}\left(S(z,\tau),z,\tau\right)y(z,t-\tau)d\tau\qquad H_1=0$$

For all these boundary types, the operators M_1 and H_1 involve time-integrals. The flow rate depends on the history of the pressure or the cross sectional area. A common approach is used for those simple, double or triple time integrals: they are divided into parts where the solution is constant in time. For example, in the impedance case where there is a double integral in time, with N the number of constant time steps in one period, and pressure p^n taken at S_n and t_n, we find:

$$\mathrm{B}_1=\left[\hat{W}_1^{n+1}\frac{\Delta t^2}{2T}\left\{\sum_{i=1}^{N-1}\tilde{p}^{n-i+1}*(y^i+y^{i+1})+y^1\tilde{p}^{n+1}+y^N\tilde{p}^{n-N+1}\right\}\right]_{z=B}\hat{n}$$

For the damped wave equations solved with Green's functions, we obtain triple integrals in time that can be treated in a similar way. The flow rate is a function of pressure history and depends also on waves coming from the far end boundary conditions and the initial conditions everywhere in the downstream domain. This thus represents more accurately the underlying physiology of pulse wave propagation.

POTENTIAL APPLICATIONS & FUTURE WORK

This new derivation will be implemented in a space-time finite element code to provide a general tool to model blood flow in part of the arterial system. The main organs and body parts outside the domain of interest can be modeled with different downstream representations in a consistent way. This approach is a step towards the modeling of the entire vascular system.

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