MATHEMATICAL FORMULATION OF A COUPLED MOMENTUM METHOD FOR MODELING BLOOD FLOW IN DEFORMABLE ARTERIES

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INTRODUCTION

In recent years, computational techniques have been used to simulate blood flow in three-dimensional models of arteries and applied to problems in disease research, device design and preoperative planning. Most of these computational analyses have only examined the velocity field (not the pressure field) and have treated the vessel walls as rigid. One of the better-known methods for including wall deformability is the ALE (arbitrarily Lagrangian-Eulerian) formulation for fluid-structure interaction problems [1]. Unfortunately, for many problems, ALE methods can be computationally expensive and, perhaps due to the continual updating of the geometry of the fluid and structural domain, not very robust.

A new approach for modeling blood flow in deformable arteries termed the *Coupled Momentum* Method *for Fluid-Solid Interaction* (CMM-FSI) is described here. The main features of this method are:

- *Strong coupling of the fluid and solid mechanics degrees of freedom.* The velocity of the vessel wall is set equal to the velocity of the fluid at the wall, since the same nodes are shared by the solids mesh and the boundary of the fluids mesh.
- *Vessel wall motion incorporated as a boundary condition for the fluid.* The elastodynamics equations governing the vessel wall motion are coupled to the conservation of mass and momentum equations governing blood flow, by assuming that the vessel wall thickness is small compared with the vessel radius.
- *Membrane formulation for the wall. B*ending is neglected and the vessel wall is approximated by a membrane.
- *Fixed fluid domain and linearized wall kinematics.* The geometry of the fluid domain and surrounding structures are fixed and the deformations of the vessel wall are accounted for through the use of linearized kinematics.

We present herein the CMM-FSI formulation and discuss how to incorporate this method into existing rigid wall finite element codes.

METHODS

Governing equations (strong form)

Fluid mechanics problem. The strong form of the continuity and momentum equations written in the advective form ([2,3]) for a domain $\Omega \in \Re^3$ is given by:

$$v_{i,t} + v_j v_{i,j} = -p_{,i}/\rho + \tau_{ij,j} + f_i \qquad (x_i, t) \in \Omega \times (0,T) \tag{1}$$

$$v_{i,i} = 0 \qquad (x_i, t) \in \Omega \times (0,T) \tag{2}$$

We complete the strong form of this fluid mechanics problem by providing suitable initial and boundary conditions. The boundary is given by (see figure 1) $\Gamma = \partial\Omega = \Gamma_g \cup \Gamma_h \cup \Gamma_s$, $\Gamma_g \cap \Gamma_h \cap \Gamma_s = \varnothing$.

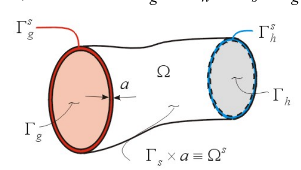

Figure 1. Boundary decomposition for the Coupled Momentum approach.

Γ_h and Γ_g represent the inlet and outlet boundaries, where Neumann and Dirichlet condition are prescribed, respectively. Γ_s represents the lateral vessel wall. While a no-slip boundary condition would be prescribed on Γ_s in the case of a rigid wall approximation, this constraint is removed in the CMM-FSI method to enable non-zero wall velocity and is replaced by the condition:

$$t_i = \sigma_{ij} n_j = t_i^f \qquad (x_i, t) \in \Gamma_s \times (0,T) \tag{3}$$

The fluid traction, t_i^f , will be specified using the elastodynamics equations for the vessel wall.

Solid mechanics problem. The classic elastodynamics equations are used to describe the vessel wall motion in a domain $\Omega^s \in \Re^3$, viz.,

$$u_{i,tt} = \sigma^s_{ij,j} + b^s_i \qquad (x_i,t) \in \Omega \times (0,T) \tag{4}$$

The initial boundary value problem for the solid mechanics problem is completed with a valid set of initial and boundary conditions.

The Coupled Momentum approach (weak form)

A new way of coupling the physics of the vessel wall and the blood flow is introduced here. This approach can be thought of as using the solid mechanics problem as a special boundary condition for the fluids problem, by relating the unknown traction t_i^f on the lateral wall of the fluid domain with the body force of the solids problem b_i^s. Womersley used an analogous approach to derive an analytical solution for pulsatile flow in an elastic vessel [4].

Fluid mechanics problem. The formulation used in this work is based on that described by Taylor *et al*. [3] and Whiting and Jansen [2]. The semi-discrete Galerkin finite element formulation produces the following weak form of the fluid mechanics problem:

$$\begin{aligned}&\int_\Omega \left\{ w_i \left(v_{i,t} + v_j v_{i,j} - f_i \right) + w_{i,j} \left(-p\delta_{ij} + \tau_{ij} \right) - q_{,i} v_i \right\} dx \\ &+ \int_{\Gamma_h} \left\{ -w_i h_i + q v_n \right\} ds \ + \int_{\Gamma_s} \left\{ -w_i t_i^f + q v_n \right\} ds = 0\end{aligned} \tag{5}$$

Solid mechanics problem. The surface traction t_i^f on the fluid lateral wall due to the wall motion is equal and opposite to the surface traction t_i^s on the vessel wall due to the fluids motion ($t_i^f = -t_i^s$). If the vessel wall thickness a is small compared with the vessel radius, this surface traction t_i^s can be considered as a body force for the solid membrane (see figure 2) using $b_i^s = t_i^s / a$.

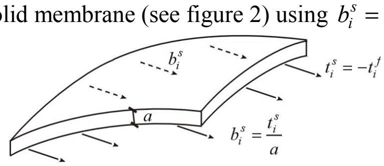

Figure 2. Relationship between the surface traction due to the fluid motion t_i^f and the solids body force b_i^s.

The weak form of the solid mechanics problem, formulated in terms of velocities, using the expression for the body force above and a linear elastic constitutive model is:

$$-\int_{\Gamma_s} w_i t_i^f ds = a\int_{\Gamma_s} w_i v_{i,t} ds + a\int_{\Gamma_s} w_{(i,j)} c_{ijkl} u_{(k,l)} ds - a\int_{\partial\Gamma_h} w_i h_i^s dl \tag{6}$$

Combined problem. If equation (6) is inserted into equation (5), we obtain the following weak form for the Navier-Stokes equations in a deformable domain:

$$\begin{aligned}&\int_\Omega \left\{ w_i \left(v_{i,t} + v_j v_{i,j} - f_i \right) + w_{i,j} \left(-p\delta_{ij} + \tau_{ij} \right) - q_{,i} v_i \right\} dx + \int_{\Gamma_h} \left\{ -w_i h_i + q v_n \right\} ds \\ &+ \int_{\Gamma_s} q v_n ds + a\int_{\Gamma_s} \left\{ w_i v_{i,t} + w_{(i,j)} c_{ijkl} u_{(k,l)} \right\} ds - a\int_{\partial\Gamma_h} w_i h_i^s dl\end{aligned} \tag{7}$$

It is important to note that the standard Galerkin method (as presented in equation (7)) is unstable for advection-dominated flows and in the diffusion dominated limit for equal-order interpolation of velocity and pressure (see Hughes *et al*. [5]). A stabilized method, as described in Taylor *et al.* [3] and Whiting and Jansen [2], is utilized to address these deficiencies of Galerkin's method. Furthermore, in order to express the displacement field u_i in the integral of the solids stresses appearing above, the Newmark family of methods is used to express this field in terms of velocities and accelerations. We do this in a manner consistent with the α - method used for the time integration of the nonlinear system of ordinary differential equations derived from the stabilized counterpart of equation (7).

Membrane formulation

The characterization of the vessel wall membrane behavior is achieved by using a new coordinate system (r,s,ζ) and triangular shape functions for the faces of the fluid tetrahedral elements lying on Γ_s (see figure 3). This new reference frame is used to compute the integrals arising from the weak form of the solid (see equations (6) and (7)).

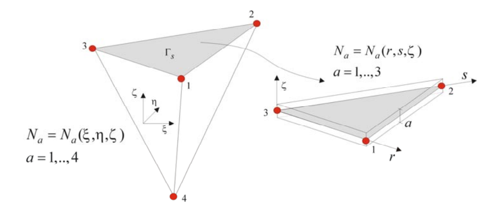

Figure 3. Diagram of a typical boundary element showing the set of shape functions used to characterize the fluids domain (left) and the solids domain (right)

CONCLUSIONS

A new method for blood flow-vessel wall interaction has been derived. The computational cost should be comparable to the rigid wall theory, since only a few new boundary integrals have to be computed. In addition, the changes with respect to a rigid wall formulation are relatively easy to incorporate into a standard rigid wall finite element program. This method could result in improved descriptions of velocity and pressure fields at relatively low cost.

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