MODELLING TWO DIMENSIONAL SOLITARY WAVES IN ARTERIES

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INTRODUCTION

The importance of understanding the mechanics of pulse wave propagation in arteries is mounting with the rising incidence of cardiovascular disease. Over the past decade the greatest effort in theoretical modelling has been placed on modelling pulses waves as solitary waves. Solitary waves are characterized by a constant speed and form of propagation in a uniform tube. A study by Yomosa [1] has firmly established the usefulness of solitary waves in modelling pulse waves in arteries.

The approximations commonly made for the numerical solution of solitary waves in arteries [2] are of three types. The first simplification affects the modelling of the artery as a shell or membrane in the large deformation regime. Invoking the "long-wave" character of the solution as justification, the axial displacement the solitary wave induces on the artery is neglected. This action unnecessarily rigidifies the elastic response of the artery, but is done to simplify the governing equations to permit certain numerical techniques to be applied more easily. The second simplification involves the numerical technique directly, which usually consists of a reductive perturbation technique [3], which results in a solitary wave solution for the first-order approximation. Invoking again the long-wave property, it is claimed that this first-order solution directly represents the exact solution. The third simplification involves the fluid model of the blood and its interaction with the artery wall. Generally, the blood is assumed to be inviscid and the mass and momentum balances are averaged over the artery diameter. The fluid-structure interaction is eventually translated into an effective normal pressure acting on the artery wall.

Although these simplifications are coherently supported by specific applications, each of these approximations deserves critical study. In previous papers [4,5], the reasons for abandoning the reductive perturbation technique were presented and an approach where the governing equations are considered directly, with no recourse to approximation, was presented. For a one-dimensional example (no axial displacements), it is shown that the wave amplitude for a given speed is obtained by simply finding the root of an algebraic equation. The shape of the solitary wave is then found through an integration of the field equations to any degree of accuracy.

This paper will describe a novel procedure for incorporating the axial displacement for the application of the direct solution approach.

GOVERNING EQUATIONS

The consideration of the axial displacement leads to a coupled system of non-linear differential equations, rather than just one equation seen in the one-dimensional case. The artery is modelled as cylindrical membrane representing a large blood vessel. The exact non-linear governing equations for a tube were derived by Epstein and Johnston [6] and are given by the following relationship,

$$[\sigma_1(1+u')/((1+u')^2 + w'^2)]' - (p/h)(1 + w/R)w' = \rho\,\ddot{u} \qquad (1)$$

$$[\sigma_1 w'/((1+u')^2 + w'^2)]' - \sigma_2/(R+w) + (p/h)(1 + w/R)(1+u') = \rho\,\ddot{w} \qquad (2)$$

where u and w are the axial and radial displacements, σ_1 and σ_2 are the axial and radial stresses, p is the normal wall pressure, h is the undeformed wall thickness, R is the undeformed artery radius and ρ is the density of the artery. The primes (') represent derivation with respect to axial position, x, and the dots (¨) represent the derivative with respect to time. The governing equations for the fluid must also be considered. A one-dimensional fluid model is again considered here, where the governing equations are given by,

$$\dot{w} + v_f w' + 1/2(R + w)v_f' = 0 \qquad (3)$$

$$\rho_f(\dot{v}_f + v_f v_f') + p' = 0 \qquad (4)$$

where v_f and ρ_f are the fluid velocity and density. This system of equations now completely defines the fluid-structure interaction that occurs as a solitary pulse wave travels through an artery.

SOLUTION APPROACH

In a typical problem, the artery is assumed to be uniformly prestressed, while the fluid moves at a constant speed $v_{f\infty}$. It is upon this background state that the solitary pulse travels. We consider a traveling wave and seek a solution of the form

$$u = u(x,t) = u(x - ct) = u(\xi) \tag{5}$$

$$w = w(x,t) = w(x - ct) = w(\xi) \tag{6}$$

where c is the wave speed and $\xi = x - ct$. We can now substitute equations (5) and (6) into equations (1), (2), (3) and (4), leaving:

$$[\sigma_1(1+u')/((1+u')^2 + w'^2)]' - (p/h)(1 + w/R)w' = \rho c u'' \tag{7}$$

$$[\sigma_1 w'/((1+u')^2 + w'^2)]' - \sigma_2/(R+w) + (p/h)(1 + w/R)(1+u') = \rho c w'' \tag{8}$$

$$(v_f - c)w' + 1/2(R + w)v_f' = 0 \tag{9}$$

$$\rho_f(v_f - c)v_f' + p' = 0 \tag{10}$$

It is a fortunate circumstance that equations (9) and (10) can be integrated exactly to obtain an explicit connection between pressure and radial displacement. A straightforward integration yields

$$p = p_\infty + (1/2)\rho_f(c - v_{f\infty})^2[1-(R/(R + w))^4] \tag{11}$$

where w_∞, p_∞, and $v_{f\infty}$ are known conditions at infinity. This rather sophisticated pressure-displacement coupling could be substituted into equations (7) and (8), producing a two equations in terms of w and u and their derivatives.

Variational Formulation

By casting this problem in a variational framework, it is possible to find explicit first integrals of the governing equations. The first integral then allows the speed, amplitude and shape of the solitary wave to be determined using the direct solution approach [6].

A variational formulation is found by searching for a Lagrangian density whose associated Euler-Lagrange equations are (7) and (8) with p given by (11). It can be verified that the function

$$L = (1/2)\rho c^2(u'^2 + w'^2) - \Sigma(u,w) + ((1 + u')(R + w)^2/2hR)[p_\infty + (1/2)\rho_f(c - v_{f\infty})^2(1 + (R/(R + w))^4)] \tag{12}$$

is the Lagrangian for the governing equations. $\Sigma(u,w)$ is the strain-energy density for the artery.

Using the derived Lagrangian, we can exploit the theorem of Noether to find first integrals for the governing equations. From Noether's Theorem, we find that two first integrals can be found from the functions

$$\partial L/\partial u' = C_1 \tag{13}$$

$$L - (\partial L/\partial u')u' - (\partial L/\partial w')w' = C_2 \tag{14}$$

By executing equations (13) and (14) on equation (12) we find that the first integrals can be represented as

$$F(u', w, w'^2) = 0 \tag{15}$$

$$G(u', w, w'^2) = 0 \tag{16}$$

The implicit function theorem allows us to eliminate u' from (15) and (16) and to write the result as a function of the form

$$w'^2 = f_c(w) \tag{17}$$

For a solitary wave solution to exist, equation (17) must have a double root at zero and single real root at $w = w_{max}$. If these roots can be found, then w_{max} is the amplitude of a solitary. The value of u'_{max} is found by substituting $w'^2 = 0$ and $w = w_{max}$ into either (15) or (16) and solving for u'. The shape of the solitary wave can then be found by simple numerical integration of the original governing equations with $w = w_{max}$ and $u' = u'_{max}$ as the initial conditions.

Numerical Example

We begin this example by selecting a representative wave speed of $c = 7$, which corresponds to the case considered by Demiray [2] and Epstein and Johnston [4]. We set the initial prestrains of the artery to 1.2 in the radial direction and 1.5 in the axial direction. The strain energy density function Σ, is chosen to be the D1 equation for arteries used by Demiray [2]. Substituting this into the governing equations and determining the variational formulation, we find a solitary wave with the shape shown in figure 1.

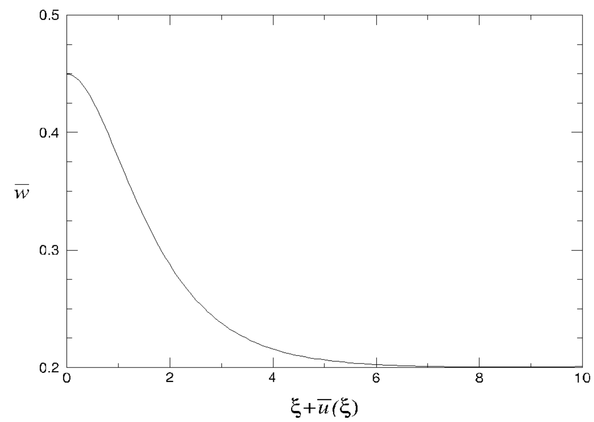

Figure 1. Calculated shape of the solitary wave

CONCLUSIONS

By considering the governing field equations directly, the speed and amplitude of solitary waves can be found by simply finding the roots of an algebraic equation. The shape of the wave is found through a simple numerical integration. This approach is also applicable in the large deformation regime. In spite of its extra complexity, the variational formulation with its attendant conserved quantities permits a solution to the exact equations to be obtained.

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