IDENTITIES FOR GENERALIZED EULER POLYNOMIALS

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Abstract. For \( N \in \mathbb{N} \), let \( T_N \) be the Chebyshev polynomial of the first kind. Expressions for the sequence of numbers \( p_i^{(N)} \), defined as the coefficients in the expansion of \( 1/T_N(1/z) \), are provided. These coefficients give formulas for the classical Euler polynomials in terms of the so-called generalized Euler polynomials. The proofs are based on a probabilistic interpretation of the generalized Euler polynomials recently given by Klebanov et al. Asymptotics of \( p_i^{(N)} \) are also provided.

1. Introduction

The Euler numbers \( E_n \), defined by the generating function

\[
\frac{1}{\cosh z} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}
\]

and the Euler polynomials \( E_n(x) \) that generalize them

\[
\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1}
\]

([2, 9.630, 9.651]) are examples of basic special functions. It follows directly from the definition that \( E_n = 0 \) for \( n \) odd. Moreover, the relation \( E_n = 2^n E_n \left( \frac{1}{2} \right) \) follows by setting \( x = \frac{1}{2} \) in (1.2), replacing \( z \) by \( 2z \) and comparing with (1.1).

Moreover, the identity

\[
\frac{2e^{xz}}{e^z + 1} = \frac{2e^{(x-1/2)z}}{e^{z/2} + e^{-z/2}}
\]

produces

\[
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} (x - \frac{1}{2})^{n-k} = \sum_{k=0}^{n} \binom{n}{k} E_k \left( \frac{1}{2} \right) (x - \frac{1}{2})^{n-k},
\]

that gives \( E_n(x) \) in terms of the Euler numbers (see [2, 9.650]).
The generalized Euler polynomials \( E_n^{(p)}(z) \), defined by the generating function
\[
\sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!} = \left( \frac{2}{1 + e^x} \right)^p e^{xz}, \quad \text{for } p \in \mathbb{N}
\]
are polynomials extending \( E_n(x) \), the case \( p = 1 \). These appear in Section 24.16 of [5]. The definition leads directly to the expression
\[
E_n^{(p)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k}^{(p)}(0),
\]
where the generalized Euler numbers \( E_n^{(p)}(0) \) are defined recursively by
\[
E_n^{(p)}(0) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(p-1)}(0) E_{n-k}(0),
\]
for \( p > 1 \) and initial condition \( E_n^{(1)}(0) = E_n(0) \).

2. A probabilistic representation of Euler polynomials and their generalizations

This section discusses probabilistic representations of the Euler polynomials and their generalizations. The results involve the expectation operator \( \mathbb{E} \) defined by
\[
\mathbb{E}g(L) = \int g(x) f_L(x) \, dx,
\]
with \( f_L(x) \) the probability density of the random variable \( L \) and for any function \( g \) such that the integral exists.

**Proposition 2.1.** Let \( L \) be a random variable with hyperbolic secant density
\[
f_L(x) = \text{sech} \, \pi x, \quad \text{for } x \in \mathbb{R}.
\]
Then the Euler polynomial is given by
\[
E_n(x) = \mathbb{E} \left( x + iL - \frac{1}{2} \right)^n.
\]

**Proof.** The right hand side of (2.3) is
\[
\mathbb{E} \left( x + iL - \frac{1}{2} \right)^n = \int_{-\infty}^{\infty} (x - \frac{1}{2} + it)^n \, \text{sech} \, \pi t \, dt
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} \left( x - \frac{1}{2} \right)^{n-j} i^j \int_{-\infty}^{\infty} t^j \, \text{sech} \, \pi t \, dt
\]
The identity
\[
\int_{-\infty}^{\infty} t^k \, \text{sech} \, \pi t \, dt = \frac{|E_k|}{2^k}
\]
holds for \( k \) odd, since both sides vanish and for \( k \) even, it appears as entry 3.523.4 in [2]. A proof of this entry may be found in [1]. Then, using \(|E_{2n}| = (-1)^n E_{2n}\) (entry 9.633 in [2])

\[
(2.5) \quad E(x + iL - \frac{1}{2})^n = \sum_{j=0}^{n} \binom{n}{j} (x - \frac{1}{2})^{n-j} \frac{E_j}{2^j} = E_n(x).
\]

There is a natural extension to the case of \( E_n^{(p)}(x) \). The proof is similar to the previous case, so it is omitted.

**Theorem 2.2.** Let \( p \in \mathbb{N} \) and \( L_j, 1 \leq j \leq p \) a collection of independent identically distributed random variables with hyperbolic secant distribution. Then

\[
(2.6) \quad E_n^{(p)}(x) = E \left[ x + \sum_{j=1}^{p} (iL_j - \frac{1}{2}) \right]^n.
\]

In a recent paper, L. B. Klebanov et al. [3] considered random sums of independent random variables of the form

\[
(2.7) \quad \frac{1}{N} \sum_{j=1}^{\mu_N} L_j
\]

where the random number of summands \( \mu_N \) is independent of the \( L_j \)'s and is described below.

**Definition 2.3.** Let \( N \in \mathbb{N} \) and \( T_N(z) \) be the Chebyshev polynomial of the first kind. The random variable \( \mu_N \) taking values in \( \mathbb{N} \), is defined by its generating function

\[
(2.8) \quad E z^{\mu_N} = \frac{1}{T_N(1/z)}.
\]

Information about the Chebyshev polynomials appears in [2] and [5].

**Example 2.4.** Take \( N = 2 \). Then \( T_2(z) = 2z^2 - 1 \) gives

\[
(2.9) \quad E z^{\mu_2} = \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{\ell=1}^{\infty} \frac{z^{2\ell}}{2^{2\ell}}.
\]

Therefore \( \mu_2 \) takes the value \( 2\ell \), with \( \ell \in \mathbb{N} \), with probability

\[
(2.10) \quad \text{Pr}(\mu_2 = 2\ell) = 2^{-\ell}.
\]

In [3], Klebanov et al. prove the following result.
Theorem 2.5 (Klebanov et al.). Assume \( \{L_j\} \) is a sequence of independent identically distributed random variables with hyperbolic secant distribution. Then, for all \( N \geq 2 \) and \( \mu_N \) defined in (2.8), the random variable
\[
L := \frac{1}{N} \sum_{j=1}^{\mu_N} L_j
\]
has the same hyperbolic secant distribution.

3. The Euler polynomials in terms of the generalized ones

The identities (1.6) and (1.7) can be used to express the generalized Euler polynomial \( E_n^{(p)}(x) \) in terms of the standard Euler polynomials \( E_n(x) \). However, to the best of our knowledge, there is no formula that allows to express \( E_n(x) \) in terms of \( E_n^{(p)}(x) \). This section presents such a formula.

Definition 3.1. Let \( N \in \mathbb{N} \). The sequence \( \{p^N_\ell : \ell = 0, 1, \ldots\} \) is defined as the coefficients in the expansion
\[
\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p^N_\ell z^\ell.
\]

Definition 2.3 shows that
\[
p^N_\ell = \Pr(\mu_N = \ell), \quad \text{for } \ell \in \mathbb{N}.
\]
The numbers \( p^N_\ell \) will be referred as the probability numbers.

Example 3.2. For \( N = 2 \), Example 2.4 gives
\[
p^{(2)}_\ell = \begin{cases} 
0 & \text{if } \ell \text{ is odd} \\
2^{-\ell/2} & \text{if } \ell \text{ is even, } \ell \neq 0.
\end{cases}
\]
The coefficients \( p^N_\ell \) are now used to produce expansions of \( E_n(x) \), one for each \( N \in \mathbb{N} \), in terms of the generalized Euler polynomials.

Theorem 3.3. The Euler polynomials satisfy, for all \( N \in \mathbb{N} \),
\[
E_n(x) = \frac{1}{N^n} \mathbb{E} \left[ E_n^{(\mu_N)} \left( \frac{1}{2} \mu_N + N(x - \frac{1}{2}) \right) \right].
\]

Proof. From (2.3) and (2.11)
\[
E_n \left( \frac{1}{2} \right) = \mathbb{E}(iL)^n = \frac{1}{N^n} \mathbb{E} \left[ \sum_{j=1}^{\mu_N} L_j \right]^n,
\]
with Theorem 2.2, this yields
\[
\mathbb{E} \left[ E_n^{(\mu_N)} \left( \frac{\mu_N}{2} \right) \right] = \mathbb{E} \left[ \sum_{j=1}^{\mu_N} L_j \right]^n = N^n E_n \left( \frac{1}{2} \right).
\]
Using identity (1.4), it follows that

\[ E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k \left( \frac{1}{2} \right) (x - \frac{1}{2})^{n-k} \]

\[ = E \left[ \sum_{k=0}^{n} \binom{n}{k} N^{-k} E_k^{(\mu_N)} \left( \frac{1}{2} \mu_N \right) (x - \frac{1}{2})^{n-k} \right] \]

\[ = E \left[ \sum_{k=0}^{n} \binom{n}{k} N^{-k} (iL_1 + \cdots + iL_{\mu_N})^k (x - \frac{1}{2})^{n-k} \right] \]

\[ = E \left[ \frac{1}{N^n} \sum_{k=0}^{n} \binom{n}{k} (iL_1 + \cdots + iL_{\mu_N})^k (N(x - \frac{1}{2})^{n-k} \right] \]

\[ = E \left[ \frac{1}{N^n} (iL_1 + \cdots + iL_{\mu_N} + N(x - \frac{1}{2}))^{n} \right] \]

\[ = E \left[ \frac{1}{N^n} (iL_1 + \cdots + iL_{\mu_N} + z - \frac{1}{2} \mu_N)^{n} \right] \]

\[ = \frac{1}{N^n} E \left[ E_n^{(\mu_N)}(z) \right], \]

where \( z = \frac{1}{2} \mu_N + N \left( x - \frac{1}{2} \right) \). This completes the proof. \( \square \)

The next result is established using the fact that the expectation operator \( E \) satisfies

\[ E[h(\mu_N)] = \sum_{k=0}^{\infty} p_k^{(N)} h(k), \]

for any function \( h \) such that the right-hand side exists.

**Corollary 3.4.** The Euler polynomials satisfy

\[ E_n(x) = \frac{1}{N^n} \sum_{k=N}^{\infty} p_k^{(N)} E_n^{(k)} \left( \frac{1}{2} k + N \left( x - \frac{1}{2} \right) \right) . \]

**Note 3.5.** Corollary 3.4 gives an infinite family of expressions for \( E_n(x) \) in terms of the generalized Euler polynomials \( E_n^{(k)}(x) \), one for each value of \( N \geq 2 \).

**Example 3.6.** The expansion (3.8) with \( N = 2 \) gives

\[ E_n(x) = \frac{1}{2^n} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_n^{(2\ell)} (\ell + 2x - 1) . \]

For instance, when \( n = 1 \),

\[ E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_1^{(2\ell)} (\ell + 2x - 1) . \]
and the value \( E_1^{(\ell)}(x) = x - \frac{\ell}{2} \) gives

\[
E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} (\ell + 2x - 1 - \ell) = x - \frac{1}{2}
\]
as expected.

4. THE PROBABILITY NUMBERS

For fixed \( N \in \mathbb{N} \), the random variable \( \mu_N \) has been defined by its moment generating function

\[
E_z^{\mu_N} = \frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell.
\]

This section presents properties of the probability numbers \( p_\ell^{(N)} \) that appear in Corollary 3.4.

For small \( N \), the coefficients \( p_\ell^{(N)} \) can be computed directly by expanding the rational function \( 1/T_N(1/z) \) in partial fractions. Example 2.4 gave the case \( N = 2 \). The cases \( N = 3 \) and \( N = 4 \) are presented below.

**Example 4.1.** For \( N = 3 \), the Chebyshev polynomial is

\[
T_3(z) = 4z^3 - 3z = 4z(z - \alpha)(z + \alpha),
\]

with \( \alpha = \sqrt{3}/2 \). This yields

\[
\frac{1}{T_3(1/z)} = \frac{z^3}{4(1 - \alpha z)(1 + \alpha z)} = \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} z^{2k+3}.
\]

It follows that \( p_\ell^{(3)} = 0 \) unless \( \ell = 2k + 3 \) and

\[
p_\ell^{(3)} = \frac{3^k}{2^{2k+2}}.
\]

Corollary 3.4 now gives

\[
E_n(x) = \frac{1}{3^n} \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} E_n^{(2k+3)}(3x + k),
\]
a companion to (3.9).

**Example 4.2.** The probability numbers for \( N = 4 \) are computed from the expression

\[
\frac{1}{T_4(1/z)} = \frac{z^4}{z^4 - 8z^2 + 8}.
\]

The factorization

\[
z^4 - 8z^2 + 8 = (z^2 - \beta)(z^2 - \gamma)
\]
with \( \beta = 2(2 + \sqrt{2}) \) and \( \gamma = 2(2 - \sqrt{2}) \) and the partial fraction decomposition
\[
\frac{z^4}{z^4 - 8z^2 + 8} = \frac{\beta}{\beta - \gamma} \frac{1}{1 - \beta/z^2} - \frac{\gamma}{\beta - \gamma} \frac{1}{1 - \gamma/z^2}
\]
show that \( p_\ell^{(4)} = 0 \) for \( \ell \) odd or \( \ell = 2 \) and
\[
p_{2\ell}^{(4)} = \frac{\sqrt{2}}{2^{2\ell+1}} \left[(2 + \sqrt{2})^{\ell-1} - (2 - \sqrt{2})^{\ell-1}\right]
\]
for \( \ell \geq 2 \). Corollary 3.4 now gives
\[
E_n(x) = \sqrt{2} \sum_{\ell=2}^{\infty} \left[\frac{(2 + \sqrt{2})^{\ell-1} - (2 - \sqrt{2})^{\ell-1}}{2^{2\ell+1}}\right] E_n^{(2\ell)}(4x + \ell - 2).
\]

Some elementary properties of the probability numbers are presented next.

**Proposition 4.3.** The probability numbers \( p^{(N)}_\ell \) vanish if \( \ell < N \).

**Proof.** The Chebyshev polynomial \( T_N(z) \) has the form \( 2^{N-1}z^N + \text{lower order terms} \). Then the expansion of \( 1/T_N(1/z) \) has a zero of order \( N \) at \( z = 0 \). This proves the statement. \( \square \)

**Proposition 4.4.** The probability numbers \( p^{(N)}_\ell \) vanish if \( \ell \not\equiv N \mod 2 \).

**Proof.** The polynomial \( T_N(z) \) has the same parity as \( N \). The same holds for the rational function \( 1/T_N(1/z) \). \( \square \)

An expression for the probability numbers is given next.

**Theorem 4.5.** Let \( N \in \mathbb{N} \) be fixed and define
\[
\theta^{(N)}_k = \frac{(2k - 1)\pi}{2N}.
\]
Then
\[
p^{(N)}_\ell = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta^{(N)}_k \cos^{\ell-1} \theta^{(N)}_k.
\]

**Proof.** The Chebyshev polynomial is defined by \( T_N(\cos \theta) = \cos(N\theta) \), so its roots are \( z_k^{(N)} = \cos \theta^{(N)}_k \), with \( \theta^{(N)}_k \) as above. The leading coefficient of \( T_N(z) \) is \( 2^{N-1} \), thus
\[
\frac{1}{T_N(z)} = \frac{2^{1-N}}{\prod_{k=1}^{N} (z - z_k)}.
\]
In the remainder of the proof, the superscript \( N \) has been dropped from \( z_k^{(N)} \) and \( \theta^{(N)}_k \), for clarity. Define
\[
Q(z) = \prod_{k=1}^{N} (z - z_k).
\]
The roots \( z_k \) of \( Q \) are distinct, therefore

\[
(4.15) \quad \frac{1}{Q(z)} = \sum_{k=1}^{N} \frac{1}{Q'(z_k)} \frac{1}{z - z_k}.
\]

The identity \( T'_N(z) = NU_{N-1}(z) \) gives

\[
Q'(z_k) = N2^{1-N}U_{N-1}(z_k)
\]

where \( U_j(z) \) is the Chebyshev polynomial of the second kind defined by

\[
U_N(\cos \theta) = \sin(N+1)\theta \sin \theta.
\]

Then

\[
(4.18) \quad U_{N-1}(z_k) = U_{N-1}(\cos \theta_k) = \sin(N\theta_k) \sin \theta_k.
\]

and the value \( \sin N\theta_k = (-1)^{k+1} \) yields

\[
(4.19) \quad Q'(z_k) = \frac{(-1)^{k+1}}{\sin \theta_k} N2^{1-N}.
\]

Therefore (4.15) now gives

\[
(4.20) \quad \frac{1}{Q(z)} = \frac{2^{N-1}}{N} \sum_{k=1}^{N} \frac{(-1)^{k+1} \sin \theta_k}{z - \cos \theta_k}.
\]

It follows that

\[
\frac{1}{T_N(1/z)} = \frac{2^{1-N}}{Q(1/z)} = \frac{1}{N} \sum_{k=1}^{N} \frac{(-1)^{k+1} z \sin \theta_k}{1 - z \cos \theta_k}
\]

\[
= \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta_k \sum_{\ell=0}^{\infty} z^{\ell+1} \cos \theta_k
\]

\[
= \frac{1}{N} \sum_{\ell=0}^{\infty} z^{\ell+1} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta_k \cos \theta_k.
\]

The proof is complete. \( \square \)

The next result provides another explicit formula for the probability numbers. The coefficients \( A(n,k) \) appear in OEIS entry A008315, as entries of the Catalan triangle.

**Theorem 4.6.** Let \( A(n,k) = \binom{n}{k} - \binom{n}{k-1} \). Then, if \( N \equiv \ell \mod 2 \),

\[
p_{\ell}^{(N)} = \frac{1}{2^{\ell}} \sum_{t=\left[\frac{\ell}{2}\right]}^{\left[\frac{\ell}{2}(2t+1)\right]} (-1)^t A(\ell - 1, \frac{1}{2}(\ell - (2t+1)N)),
\]
indent when \( \ell \) is not an odd multiple of \( N \) and

\[
p_{\ell}^{(N)} = \frac{1}{2^l} \left[ \sum_{s=1}^{\left\lfloor \frac{\ell}{N} - 1 \right\rfloor} (-1)^{k-s} A(\ell - 1, sN) \right] + \frac{(-1)^k}{2^{l-1}}, \quad \text{with } k = \frac{1}{2} (\ell/N - 1)
\]

otherwise.

The proof begins with a preliminary result.

**Lemma 4.7.** Let \( N \in \mathbb{N} \) and \( \theta_k = \frac{\pi}{2} \left( \frac{2k-1}{N} \right) \). Then

\[
f_N(z) = \sum_{k=1}^{N} (-1)^{k+1} e^{i\theta_k z}
\]

is given by

\[
f_N(z) = \frac{1 - (-1)^N e^{\pi iz}}{2 \cos \left( \frac{\pi z}{2N} \right)} \quad \text{if } z \neq (2t + 1)N \text{ with } t \in \mathbb{Z}
\]

and

\[
f_N(z) = (-1)^t N t \quad \text{if } z = (2t + 1)N \text{ for some } t \in \mathbb{Z}.
\]

In particular

\[
f_N(k) = \begin{cases} 
(-1)^{(k/N-1)/2} N t & \text{if } k/N \text{ is an odd integer} \\
1 - (-1)^N & 2 \cos \left( \frac{\pi z}{2N} \right) \text{ otherwise.}
\end{cases}
\]

**Proof.** The function \( f_N \) is the sum of a geometric progression. The formula (4.22) comes from (4.24) by passing to the limit. \( \square \)

The proof of Theorem 4.6 is given now.

**Proof.** The expression for \( p_{\ell}^{(N)} \) given in Theorem 4.5 yields

\[
p_{\ell}^{(N)} = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \left( e^{i\theta_k} - e^{-i\theta_k} \right) e^{i\theta_k} = \frac{1}{2^l N i} \sum_{k=1}^{N} (-1)^{k+1} \sum_{r=0}^{\ell-1} \binom{\ell - 1}{r} \left[ e^{i(\ell-2r)\theta_k} - e^{i(\ell-2r-2)\theta_k} \right]
\]

\[
= \frac{1}{2^l N i} \sum_{r=0}^{\ell-1} \binom{\ell - 1}{r} \left[ f_N(l - 2r) - f_N(l - 2r - 2) \right]
\]

\[
= \frac{1}{2^l N i} \left[ \sum_{r=1}^{\ell-1} A(\ell - 1, r) f_N(\ell - 2r) + f_N(\ell) - f_N(-\ell) \right].
\]

Now \( f_N(\ell) = f_N(-\ell) = 0 \) if \( \ell/N \) is not an odd integer. On the other hand, if \( \ell = (2t + 1)N \), with \( t \in \mathbb{Z} \), then

\[
f_N(\ell) = (-1)^t N t \text{ and } f_N(-\ell) = -(-1)^t N t.
\]
Thus
\[ f_N(\ell) - f_N(-\ell) = \begin{cases} 2Nt(-1)^{(\ell/N-1)/2} & \text{if } \ell \text{ is an odd multiple of } N \\ 0 & \text{otherwise} \end{cases} \]

The simplification of the previous expression for \( p^{(N)}_\ell \) is divided in two cases, according to whether \( \ell \) is an odd multiple of \( N \) or not.

**Case 1.** Assume \( \ell \) is not an odd multiple of \( N \). Then

\begin{equation}
(4.26) \quad p^{(N)}_\ell = \frac{1}{2tN} \sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell-2r).
\end{equation}

Moreover,

\begin{equation}
(4.27) \quad f_N(\ell-2r) = \begin{cases} (-1)^t Nt & \text{if } \frac{\ell-2r}{N} = 2t+1 \\ 0 & \text{otherwise} \end{cases}
\end{equation}

Therefore

\begin{equation}
(4.28) \quad p^{(N)}_\ell = \frac{1}{2t} \sum_{t=\frac{1}{2}(\frac{\ell}{N} - 1)}^{\frac{1}{2}(\frac{\ell}{N} - 1)} (-1)^t A(\ell-1, r).
\end{equation}

Observe that \( \ell - (2t+1)N \) is always an even integer, thus the index \( r \) may be eliminated from the previous expression to obtain

\begin{equation}
(4.29) \quad p^{(N)}_\ell = \frac{1}{2t} \sum_{t=\frac{1}{2}(\frac{\ell}{N} - 1)}^{\frac{1}{2}(\frac{\ell}{N} - 1)} (-1)^t A(\ell - 1, \frac{1}{2}(\ell - (2t+1)N)).
\end{equation}

**Case 2.** Assume \( \ell \) is an odd multiple of \( N \), say \( \ell = (2k+1)N \). Then

\begin{align*}
p^{(N)}_\ell &= \frac{1}{2tN} \left[ \sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell-2r) + 2Nt(-1)^k \right] \\
&= \frac{1}{2tN} \left[ \sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell-2r) \right] + \frac{(-1)^k}{2^{\ell-1}}.
\end{align*}

The term \( f_N(\ell-2r) \) vanishes unless \( \ell - 2r \) is an odd multiple of \( N \). Given that \( \ell = (2k+1)N \), the term is non-zero provided \( 2r \) is an even multiple of \( N \); say \( r = sN \) for \( s \in \mathbb{N} \). The range of \( s \) is \( 1 \leq s \leq \frac{\ell}{2N} = 2k+1 - \frac{1}{N} \). This implies \( 1 \leq s \leq 2k = \ell/N - 1 \), and it follows that

\begin{align*}
p^{(N)}_\ell &= \frac{1}{2t} \left[ \sum_{s=1}^{\ell/N-1} (-1)^{k-s} A(\ell-1, sN) \right] + \frac{(-1)^k}{2^{\ell-1}}, \text{ with } k = \frac{1}{2} (\ell/N - 1).
\end{align*}

The proof is complete. \( \square \)
Note 4.8. The expression in Theorem 4.6 shows that $p^{(N)}_\ell$ is a rational number with a denominator a power of 2 of exponent at most $\ell$. Arithmetic properties of these coefficients will be described in a future publication [4]. Moreover, the probability numbers $p^{(N)}_\ell$ appear in the description of a random walk on $N$ sites. Details will appear in [4].

5. AN ASYMPTOTIC EXPANSION

The final result deals with the asymptotic behavior of the probability numbers $p^{(N)}_\ell$.

Theorem 5.1. Let $\varphi_N(z) = \mathbb{E}[z^{N\mu}]$. Then, for fixed $z$ in the unit disk $|z| < 1$,

$$\varphi_N(z) \sim \left(\frac{z}{1 + \sqrt{1 - z^2}}\right)^N,$$

as $N \to \infty$.

Proof. The generating function satisfies

$$\varphi_N(z) = \frac{1}{T_N(1/z)} = \frac{z^N}{2^{N-1}} \prod_{k=1}^{N} \left(1 - z \cos \theta_k^{(N)}\right)^{-1},$$

with $\theta_k^{(N)} = (2k - 1)\pi/2N$ as before. Then

$$\log \varphi_N(z) = \log 2 + \frac{z}{2} - \sum_{k=1}^{N} \log \left(1 - z \cos \theta_k^{(N)}\right).$$

The last sum is approximated by a Riemann integral

$$\frac{1}{N} \sum_{k=1}^{N} \log \left(1 - z \cos \theta_k^{(N)}\right) \sim \frac{1}{\pi} \int_{0}^{\pi} \log(1-z\cos \theta) \, d\theta = \log \left(\frac{1 + \sqrt{1 - z^2}}{2}\right).$$

The last evaluation is elementary. It appears as entry 4.224.9 in [2]. It follows that

$$\log \varphi_N(z) \sim \log 2 + N \log \left(\frac{z}{2}\right) - N \log \left(\frac{1 + \sqrt{1 - z^2}}{2}\right) + \text{indent}$$

and this is equivalent to the result. $\square$

The function

$$A(z) = \frac{2}{1 + \sqrt{1 - 4z}} = \sum_{n=0}^{\infty} C_n z^n$$

is the generating function for the Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$
The final result follows directly from the expansion of Binet’s formula for Chebyshev polynomial

$$T_N(z) = \frac{(z - \sqrt{z^2 - 1})^N + (z + \sqrt{z^2 - 1})^N}{2}.$$  

Some standard notation is recalled. Given two sequences $a = \{a_n\}$, $b = \{b_n\}$, their convolution $c = a * b$ is the sequence $c = \{c_n\}$, with

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$ 

The convolution power $c^{(sN)}$ is the convolution of $c$ with itself, $N$ times.

**Theorem 5.2.** For $N \in \mathbb{N}$ fixed, the first $N$ nonzero terms of the sequence $q_{\ell}^{(N)} = 2^{\ell-1} p_{\ell}^{(N)}$ agree with the first $N$ terms of the $N$-th convolution power $C_n^{(sN)}$ of the Catalan sequence:

$$q_0^{(N)} = C_0^{(sN)}, \quad q_{N+2}^{(N)} = C_1^{(sN)}, \quad \ldots, \quad q_{N+2k}^{(N)} = C_k^{(sN)}, \quad \ldots, \quad q_{3N-2}^{(N)} = C_{N-1}^{(sN)}.$$ 

In terms of generating functions, this is equivalent to

$$\left( \sum_{n=0}^{\infty} C_n z^{2n+1} \right)^N - \sum_{\ell=0}^{\infty} q_{\ell}^{(N)} z^\ell \sim 2^N z^{3N}.$$ 

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**References**


