Homework V: Problems Concerning the Accretion and Layering of the Earth

1. The gravitational potential is the energy required to move a unit mass from a point p, which is at a distance, r, from a point mass, M, to infinity. It can also be thought of as the Energy released when a unit mass falls from infinity to the point p.

   Recalling that energy, U, is given by

   $$U = -\int F dS \quad \text{(force times displacement)}$$

   If the force acting on this unit mass is $F = \frac{GM}{r^2}$, show that the gravitational potential energy released by the unit mass falling to the surface of a body with mass M is given by

   $$U = GM/r$$

2. Suppose that the Earth (total mass, M=5.98 x 10^{24} kg) formed by accretion of particles of constant density falling in from infinity.

   (a) Using the equation in 1 above, assuming a uniform density of the growing Earth, show that the total energy released by the accretion process for the entire Earth, E is:

   $$E = \frac{3G M_E^2}{5R}$$

   (Hint: consider the stage of the accretion process when the radius of the earth is r. What is the gravitational potential per unit mass at its surface? Add a mass dm in a shell of thickness dr. How much energy (dE) is released in this process? Note that U = dE/dm, and $m = 4/3\pi r^3 \rho$.)

   b) Using the equation derived above, calculate the total amount of energy released during the accretion of the Earth.

   c) Given the average heat capacity of Earth materials is about 1000 joules K^{-1} kg^{-1} and the average heat of fusion of this material is 5 x 10^5 joules kg^{-1}, how much heat would be necessary to raise the temperature of the accreting body from 0 K to a melting point of 2000 K and then melt the body?

   d) Is the amount of heat released by the accretion process, as calculated above, sufficient to melt the Earth in the case above?

   e) Assume that all of the heat generated by accretion would only go into raising the temperature of the Earth, (i.e. ignore the heat of fusion), and that 50% of the accretional heat is radiated back into space. Starting at an initial temperature of

100 K, what would the final temperature be after the entire mass of the Earth had accreted?

3. Assume that the primitive Earth was a homogeneous liquid sphere with density, ρ and radius R as follows:

$\rho = 5{,}520$ kg/m^3 and $R = 6.37 \times 10^6$ m,

and that this molten ball behaved rheologically as a perfect fluid (i.e., no tensile strength and hence no resistance to deformation). If the period of rotation (or alternatively the length of a day) was 1 hr. 40 min., would spallation have occurred wherein globs of molten material were thrown off the Earth in its equatorial plane? To answer this question we must keep in mind the following:

The geopotential (U_{GP}) may be thought of as the sum of two terms:

1) the gravitational potential (U_g) which causes a particle of unit mass to be accelerated towards the center of a uniform sphere, and

2) centrifugal potential (U_c) which accelerates a unit mass directly away from the rotation axis.

If the Earth is assumed to be symmetrical about its rotation axis and we include only terms of order 2, then the gravitational potential (U_g) can be expressed as:

$$U_g = GM/r - GMR^2/2r^3 \ [J_2 (3\text{Sin}^2\phi - 1)]$$

where

$G = 6.67 \times 10^{-11}$ m^3/kg s^2

R = the mean radius

ϕ = the latitude

J_2 = a coefficient related to the way mass is distributed in the Earth.

The centrifugal potential is given by:

$$U_c = \omega^2 r^2 \cos^2\phi / 2$$

where ω = the angular velocity.

a) At the equator ($\phi = 0$) the gravitational and centrifugal accelerations act in opposite directions. Noting that

$$-g = dU_{GP} /dr,$$

what must be true of the magnitude of g at a point of critical instability (i.e., where the sum of the forces acting on a mass is zero)?

b) Assuming $J_2 = 0$, calculate the angular velocity and rotation rate at critical instability. Note that the rotation rate should be expressed in units of minutes/day, where a day is one revolution, and one revolution is 2π radians.

c) What angular velocity does a 100 minute day correspond to?

d) To answer the question initially proposed, is the angular velocity for a 100 min. day fast enough under these conditions ($J_2 = 0$) to cause mass to be shed?

e) What is the effect of a positive J2? Does it enhance or reduce the likelihood of spallation?

4. It has been hypothesized that the moon was one time a part of the Earth and that it was shed from the Earth by spallation. We want to examine two possibilities, using the results calculated in problem 3.

a. Consider that in the Earth a chemical separation occurred to form a mantle and core, with a core radius of 3.47×10^6m and densities of 4,180 kg/m^3 and 11,670 kg/m^3, for the mantle and core respectively. Using the principle of conservation of angular momentum, what is the new rotation rate after separation?

Is this greater than or less than the critical rotation rate for instability?

To answer this, let H = angular momentum. For conservation of angular momentum

$$H_i = H_f$$

where i and f refer to initial and final states, respectively.

Also,

$$H = I\omega$$

where I = the moment of inertia about the rotation axis.

Recall that for a uniform sphere

$$I = 2/5\ MR^2$$

and you derived an equation to calculate the moment of inertia for a layered body back in Homework 3.

$$I = 8\pi/15[\rho_c r_c^5 + \rho_m(r_a^5 - r_c^5)]$$

b. Calculate the rotation rate after core separation, as above, except use the value of the moment of inertia of the differentiated Earth, $I = 0.3308 M_E R_E^2$. Is this greater than or less than the critical rotation rate for instability?

c. What is the critical assumption we are making in these calculations, i.e. what factor have we assumed, and how could we know if our assumption is correct?