THE WEDGE PRODUCT FOR SOPHOMORE CALCULUS

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The exterior algebra offers a computational and conceptual tool which can be introduced in sophomore multivariable calculus with a minimum of formalism. The purpose of this note is to demonstrate how that can be done for the ordinary second year calculus student from the very beginning of his or her study.

Although the formulations in this paper have been carried out in n-dimensional Euclidean space, which might strike some readers as notationally forbidding, each proposition can be stated and proved in three our four dimensions to avoid this notational generality, and except for notation, all statements and proofs will not differ from the n-dimensional version.

When the exterior algebra is available from the outset in a sophomore calculus course, it can be used to discuss k-dimensional planes, simultaneous linear systems of equations, linear transformations and all aspects of multivariable integration, the gradient, divergence and curl culminating in a single Stokes' theorem for differential forms which subsumes all of the separate Green, Gauss, divergent and classical Stokes results.

Cartan originally introduced the exterior algebra in order to simplify calculations with integrals. Not only are calculations simplified, but also concepts become unified, geometric, and easy to remember. After fifty years it is surprising not to find these techniques incorporated into the standard calculus curriculum at an early stage.

In this paper we will presume the standard introduction to vectors and the inner (dot) product which appears in every multivariable calculus text.

1. Wedge Product

Definition 1. Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ be k vectors in $\mathbb{R}^n$. Define the wedge product $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k$ by stipulating that

(i) $\underline{v} \wedge \underline{v} = 0$ for any vector $\underline{v} \in \mathbb{R}^n$.

(ii) $\underline{v} \wedge \underline{w} = (-1)\underline{w} \wedge \underline{v}$ for any vectors $\underline{v}$ and $\underline{w}$ in $\mathbb{R}^n$.

(iii) With the exception of (i) and (ii) all algebraic rules which apply to "ordinary multiplication", also apply to "$\wedge$".

Let $\underline{e}_k = (0, \ldots, 0, 1, 0 \ldots 0) \in \mathbb{R}^n$ and let $I_n = \{1, 2, \ldots, n\}$.

$$\begin{aligned}
\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k &= \left(\sum_{i=1}^{n} a_{i1}\, \underline{e}_i\right) \wedge \left(\sum_{i=1}^{n} a_{i2}\, \underline{e}_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} a_{ik}\, \underline{e}_i\right) \\
&= \sum_{(i_1, i_2, \ldots, i_k) \in I_n^k} a_{i_1 1}, a_{i_2 2}, a_{i_3 3} \ldots a_{i_n k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k} \\
&= \sum_{\substack{(i_1, i_2, \ldots, i_k) \in I_n^k \\ i_\alpha \neq i_\beta; 1 \leq \alpha, \beta \leq k}} a_{i_1 1}, a_{i_2 2} \ldots a_{i_n k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}\ , \\
&\qquad \text{by applying (i),} \\
&= \sum_{\substack{i_1 < i_2 < \cdots < i_k \\ (i_1, i_2, \ldots, i_k) \in I_k^n}} (-1)^\gamma a_{i_1 1}, a_{i_2 2} \ldots a_{i_n k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}\ , \\
&\qquad \text{by applying (ii),}
\end{aligned}$$

where γ denotes the number of transpositions required to obtain $i_1 < i_2 < \cdots < i_k$. Any k-fold wedge product can be expressed in this reduced form.

Note: This should be demonstrated concretely in $\mathbb{R}^3$ or $\mathbb{R}^4$. Students, without mastering the above formalism, can adapt without difficulty to putting the wedge product of vectors in $\mathbb{R}^n$ into reduced form.

Definition 2. In reduced form $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_n = c\,\underline{e}_1 \wedge \underline{e}_2 \wedge \cdots \wedge \underline{e}_n$ and c is the determinant of the matrix,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

where $\underline{v}_j = \sum_{i=1}^{n} a_{ij}\underline{e}_i$, for $j = 1, 2, \dots, n$. Restating,

$$c = \det(a_{ij}) \ , \quad 1 \leq i, j \leq n \ .$$

This definition is stated because the usual sophomore is unfamiliar with determinants larger than 3×3. Though inefficient, determinants so defined, can be computed in higher dimensions using this definition.

Definition 3. Define $\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\|^2 = \det \begin{pmatrix} \underline{v}_1 \cdot \underline{v}_1 & \dots & \underline{v}_1 \cdot \underline{v}_k \\ \underline{v}_2 \cdot \underline{v}_1 & \dots & \underline{v}_2 \cdot \underline{v}_k \\ \vdots & & \vdots \\ \underline{v}_k \cdot \underline{v}_1 & \dots & \underline{v}_k \cdot \underline{v}_k \end{pmatrix}$.

$\|\underline{v}_1 \wedge \underline{v}_2 \wedge \dots \underline{v}_n\|$ is called *the length* of the wedge product $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k$.

Agreement of this definition with the usual definition of length $\|\underline{v}\|$ of a vector $\underline{v}$ in $\mathbb{R}^3$ follows from $\|\underline{v}\|^2 = \underline{v} \cdot \underline{v}$.

Definition 4. Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k \in \mathbb{R}^n$. And define

$$\mathcal{P}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n) = \left\{ \sum_{i=1}^{k} c_i \underline{v}_i : 0 \leq c_i \leq 1 \ , \ c_i \in \mathbb{R} \right\} \ .$$

$\mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n)$ is called *the parallelipiped determinant* by $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$. Again a demonstration of the definition of $\mathbb{R}^2$ and $\mathbb{R}^3$ motivates the student.

2. Fundamental Facts about the Wedge Product

Theorem 1. For $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in \mathbb{R}^n$,

$$\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\| = \mathrm{vol}_k \mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$$

where vol_k is understood traditionally as volume of the $(k-1)$ dimensional base times the altitude.

Proof. Proceed by induction on k. The $k = 1$ case is trivial. Now to show the $(k-1)$ case implies the k case, write

$$\underline{v}_k = \underline{x} + \sum_{i=1}^{k-1} c_i v_i \quad \text{where } \underline{x} \perp \underline{v}_i \ , i = 1, 2, \ldots, (k-1) \ .$$

Then $\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\|^2 = \|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{k-1} \wedge \underline{x}\|^2 = \left(\det(\underline{v}_i \cdot \underline{v}_j)_{1 \le i,j \le k-1}\right) (\underline{x} \cdot \underline{x}) = \mathrm{vol}^2_{\mathrm{k-1}} \mathcal{P}(\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \ldots, \underline{\mathrm{v}}_{\mathrm{k-1}}) \|\underline{\mathrm{x}}\|^2$. Therefore,

$$\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\| = \mathrm{vol}_k \mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{k-1}, \underline{v}_k) \ .$$

Theorem 2. For $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in \mathbb{R}^n$,

$$\|\underline{v}_1, \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\|^2 = \sum a^2_{i_1, i_2 \ldots i_k}$$

where $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k = \sum\limits_{\substack{i_1 < i_2 < \cdots < i_k \\ 1 \le i,j \le n}} a_{i_1, i_2 \ldots i_k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}$.

Remark. This theorem provides the student with an easier way to compute $\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\|$ coinciding with the way vector lengths are calculated. Simply put the wedge product into reduced form and take the square root of the sum of the squares of the coefficients.

Proof. The proof is by induction using the reduction of the previous proof.

$$\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{k-1}\|^2 = \sum_{\substack{i_1<i_2<\cdots<i_{k-1} \\ 1\leq i,j\leq n}} a_{i_1,i_2\ldots i_{k-1}} = \det\left(\underline{v}_i, \underline{v}_j\right)_{1\leq i,j\leq k-1} .$$

But $\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\|^2 = \det(\underline{v}_i \cdot \underline{v}_j)_{1\leq i,j\leq k} = \left(\det(\underline{v}_i \cdot \underline{v}_j)_{1\leq i,j\leq k-1}\right)(x \cdot x) = \sum_{\substack{i_1<i_2<\cdots<i_k \\ 1\leq i,j\leq n}} a^2_{i_1,i_2\ldots i_k}$. The $k = 1$ case is trivial.

Theorem 3. For $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in \mathbb{R}^n$,

$$\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k = 0 \text{ if and only if } \underline{v}_j = \sum_{\substack{i=1 \\ i\neq j}} a_i \underline{v}_i \text{ for some } j, 1 \leq j \leq k .$$

Proof. (if) $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k = \underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \sum_{\substack{i=1 \\ i\neq j}}^{k} a_i \underline{v}_i \wedge \cdots \wedge \underline{v}_k = 0$.

(only if) $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k = 0$ implies by Theorem 1 that $\mathrm{vol}_k \mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k) = 0$. Hence one of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ is a linear combination of the others.

N.B. "$\frown$" indicates that symbol is omitted.

Definition 4. The $*$-operator on an $(n-1)$-fold wedge product $\mathbb{R}^n$ is defined as follows:

(i) $*$-operator is linear.

(ii) $*(\underline{e}_1 \wedge \underline{e}_2 \wedge \cdots \wedge \widehat{\underline{e}}_j \wedge \ldots \bar{\underline{e}}_n) = (-1)^{j-1}\underline{e}_j$

The $*$-operator on an n-fold wedge product in $\mathbb{R}^n$ is defined as follows:

(i) $*$-operator is linear.

(ii) $*(\underline{e}_1 \wedge \cdots \wedge \underline{e}_n) = 1$.

Theorem 5. For $\underline{w}_1, \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{n-1} \in \mathbb{R}^n$,

$$(\underline{w} \wedge \underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1}) = \underline{w} \cdot * \underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1} \ .$$

Comment. Theorem 5 can be used to reinforce the role of the wedge product in computing volumes of parallelepipeds, since

$$\begin{aligned} \| * \underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_n \| &= \| \underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_n \| \\ &= \left| \frac{\underline{v}}{\|\underline{w}\|} \cdot v_n \right| \|\underline{w}\| \\ &= \text{Volume of the base} \times \text{altitude.} \end{aligned}$$

Proof. If $\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1} = \sum_{j=1}^{n} a_{12\ldots\widehat{j}\ldots n} \underline{e}_1 \wedge \underline{e}_2 \wedge \cdots \wedge \widehat{\underline{e}}_j \wedge \ldots \underline{e}_n$, and $\underline{w} = \sum_{j=1}^{n} w_i \underline{e}_i$, then

$$* \left(\sum_{j=1}^{n} a_{12\ldots\widehat{j}\ldots n} \underline{e}_1 \wedge \underline{e}_2 \wedge \cdots \wedge \widehat{\underline{e}}_j \wedge \ldots \underline{e}_n \right) \cdot \underline{w} =$$

$$\left(\sum_{j=1}^{n} (-1)^{j-1} a_{12\ldots\widehat{j}\ldots n} \underline{e}_j \right) \cdot \left(\sum_{j=1}^{n} w_j \underline{e}_j \right) = \sum_{j=1}^{n} (-1)^{j-1} a_{12\ldots\widehat{j}\ldots n} w_j \ . \text{ And}$$

$$* \left(\underline{w} \wedge \underline{v}_1 \wedge \cdots \wedge \underline{v}_k \right) = * \left(\underline{w} \wedge \sum_{j=1}^{n} a_{12\ldots\widehat{j}\ldots n} \underline{e}_1 \wedge \cdots \wedge \widehat{\underline{e}}_j \wedge \ldots \underline{e}_n \right) = \sum_{j=1}^{n} (-1)^{j-1} a_{12\ldots\widehat{j}\ldots n} w_j \ .$$

Theorem 6. For $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{n-1} \in \mathbb{R}^n$, $*\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1}$ is a vector $\underline{w}$ satisfying

(i) $\underline{w} \cdot \underline{v}_j = 0$ ($\underline{w} \perp \underline{v}_j$) for $1 \leq j \leq n-1$.

(ii) $\|\underline{w}\| = \mathrm{vol_{n-1}}\mathcal{P}(\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \ldots, \underline{\mathrm{v}}_{\mathrm{n}-1})$.

Proof. $\underline{w} \cdot \underline{v}_i = (*\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1}) \cdot \underline{v}_i = *\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1} \wedge \underline{v}_i$ for $i = 1, 2, \ldots, (n-1)$ by Theorem 5. But $*\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1} \wedge \underline{v}_i = 0$. And for (ii), simply note that $\|*\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1}\| = \|*\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_{n-1}\|$, by Theorem 2. But $\|\underline{v}_1 \wedge \underline{v}_2 \cdots \wedge \underline{v}_{n-1}\| = \mathrm{vol_{n-1}}\mathcal{P}(\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \ldots, \underline{\mathrm{v}}_{\mathrm{n}-1})$.

Theorem 7. If $T : \mathbb{R}^n$, $m \leq n$, is a linear map, then the change in volume of the unit parallelepiped $\mathcal{P}(\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n)$ under the mapping T is $\|T(\underline{e}_1) \wedge \cdots \wedge T(\underline{e}_n)\| = \det T$.

Proof.

$$\begin{aligned}\|T(\underline{e}_1) \wedge \cdots \wedge T(\underline{e}_n)\| &= \mathrm{vol_n}\mathcal{P}\left(\mathrm{T}(\underline{\mathrm{e}}_1), \mathrm{T}(\underline{\mathrm{e}}_2), \ldots, \mathrm{T}(\underline{\mathrm{e}}_\mathrm{n})\right) \\ &= \det T \ \mathrm{vol_n}\mathcal{P}(\underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2, \ldots, \underline{\mathrm{e}}_\mathrm{n})\end{aligned}$$

assuming that the student is aware of matricial representations of linear transformations by matrices whose columns are the coordinates of the image of the standing basis vectors $\underline{e}_2, \ldots \underline{e}_n$.

Remarks. In $\mathbb{R}^3$,

1. $\underline{v} \times \underline{w} = *\underline{v} \wedge \underline{w}$

2. $\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{u} \cdot *\underline{v} \wedge \underline{w}) = \underline{u} \wedge \underline{v} \wedge \underline{w}$

$$\begin{aligned}&=(u_1\underline{e}_1 + u_2\underline{e}_2 + u_3 e_3) \wedge (v_1\underline{e}_1 + v_2\underline{e}_2 + v_3 e_3) \wedge (w_1\underline{e}_1 + w_2\underline{e}_2 + w_3\underline{e}_3) \\ &= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.\end{aligned}$$